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The Cauchy Integral Formula for Biregular Function in Octonionic Analysis

Yonghua Guo*, Haiyan Wang

School of Science, Tianjin University of Technology and Education, Tianjin, China

Email address:
1974184907@qq.com (Yonghua Guo)
*Corresponding author

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Abstract: In this paper, we mainly study the Cauchy integral formula and mean value theorem for biregular function in octonionic analysis. Octonion is the extension of complex number to non-commutative and non-associative space. Because of the non-associative properties of multiplication, octonion plays an important role in wave equation, Yang-Mills equations, operator theory and so on. In recent years, octonion has become a hot topic for scholars at home and abroad and got many rich results, such as Fourier transform, Bergman kernel, Taylor series and its applications in quantum mechanics. On the basis of two Stokes theorems, we get Cauchy integral formula for biregular function in octonionic analysis by using the methods in dealing with the Cauchy integral formula for biregular function in Clifford analysis and regular function in octonionic analysis. As a direct result we also get the mean value theorem for biregular function in octonionic analysis. This will generalize the corresponding conclusion in complex analysis and Clifford analysis, and lays a solid foundation for the application of octonionic analysis in physics.

Keywords: Octonion, Biregular Function, Cauchy Integral Formula, Mean Value Theorem

1. Introduction

Octonion is a generalization of quaternion to nonassociative algebra which has played a very important role in physical phenomena of black hole [1], supersymmetry and duality, extended supersymmetry [2], supergravity models etc [3-7]. Wang wei professor and his collaborators discussed the octonion Heisenber group [8]. Calin, O. and Chang, D. C. and Markina, I studied the geometric analysis on H-type groups related to division algebras [9]. In 1998, X. M. Li studied octonionic analysis [10]. Octonion has been widely studied by Baez [11]. Recently, Many experts and scholars are dedicated to octonionic analysis and obtain some results such as Cauchy integral formula for regular function, Hardy space, Bergman space [12-15]. H. Y. Wang and his collaborators studied the right inverse of Dirac in octonion space and generalized octonionic analysis to octonionic analysis of several variables [16-17]. J. X. Wang, X. M. Li described the octonion Bergman kernel for the unit ball [18]. In this paper, we will study regular function of two variables, called biregular function. But the octonions are neither commutative nor associative, which bring barriers to the study of the problems in biregular function. Therefore, we have biregular Cauchy integral formula and mean value theorem of octonions by use the associator to overcome the difficulties.

2. Octonion

The octonions O are the nonassociative, noncommutative, normed division algebra over the real generated by $e_1,\dots,e_7$ [1], where the multiplication rules between the basis are given as follows [2, 10]:

$$e_i e_j + e_j e_i = -2\delta_{ij}, i, j = 1,\dots,7,$$

$$e_0^2 = e_0 = 1,\ e_i e_0 = e_0 e_i = e_i,\ e_i^2 = -1, i = 1,\dots,7. \tag{1}$$

For each $x \in O$, it can be written a $x = \sum_{i=0}^{7} x_i e_i \, (x_i \in \mathbb{R})$ and its conjugate

$$\overline{x} = x_0 e_0 - \sum_{i=1}^{7} x_i e_i, \tag{2}$$

where $\overline{e_0} = e_0$, $\overline{e_j} = -e_j (j = 1,...,7)$. Then $\overline{e_i e_j} = \overline{e_j}\,\overline{e_i} (\forall i, j = 1,...,7)$ and $\overline{xy} = \overline{y}\,\overline{x} (x, y \in O)$.

In order to illustrate the relationship between octonion multiplication, shown in Figure 1. Fano mnemonic where for simplicity we have used which consists of 7-points and 7-directed lines, then 7-points represent the standard basis for octonions.

Octonion multiplication are the nonassociative, noncommutative. But the subalgebra generated by any two elements is associative, namely, the octonions are alternative. So, for any $x, y, z \in O$, the associator [x, y, z] is defined by [x, y, z]=(xy)z−x (yz).

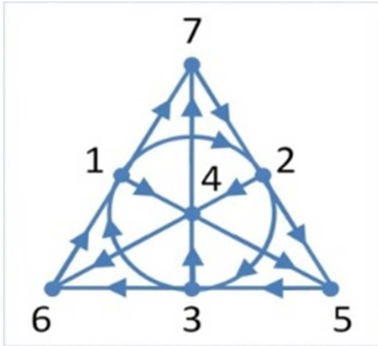

Figure 1. *Fano Mnemonic.*

Octonions obey some weakened associative laws, including the so-called Moufang identities, for any $x, y, z \in O$, it satisfies [1, 19].

$$[x, y, z] = -[y, x, z] = [y, z, x], [x, x, y] = \left[x, \overline{x}, y\right] = 0.\ (xyx)z = x\left(y(xz)\right), z(xyx) = \left((zx)y\right)x, x(yz)x = (xy)(zx). \tag{3}$$

3. Biregular Function

The corresponding Dirac operator is defined as

$$D = \sum_{i=0}^{7} e_i \partial_{x_i}. \tag{4}$$

More precisely,

$$Df(x) = \sum_{i=0}^{7}\sum_{j=0}^{7} e_i e_j \partial_{x_i} f_j, f(x)D = \sum_{i=0}^{7}\sum_{j=0}^{7} e_j e_i \partial_{x_i} f_j, \tag{5}$$

for any O-valued function $f(x) = \sum_{j=0}^{7} e_j f_j(x)$ with $f_j(x)$ being real valued.

Definition 1: Let $\Omega \subset \mathbb{R}^8$ be a nonempty, open and connected set and $f \in C^1(\Omega, O)$. If for any $x \in \Omega$ we have

$$Df(x) = 0, (f(x)D = 0) \quad \text{in } \Omega, \tag{6}$$

then the function is said to be left regular function (right regular function) in Ω. In short, left regular function is also called regular function.

Definition 2: Suppose $\Omega = \Omega_1 \times \Omega_2$ be a nonempty, open and connected set in $\mathbb{R}^8 \times \mathbb{R}^8$ and $f \in C^1(\Omega, O)$. If for any $(x, y) \in \Omega$ we have

$$\begin{cases} D_x f(x, y) = 0 \\ f(x, y) D_y = 0 \end{cases} \tag{7}$$

then the function f is called as a biregular function in Ω.

4. Cauchy Integral Formula

We introduce the Cauchy kernel, which satisfies the relation

$$E_1(u - x) = \frac{1}{w_8}\frac{\overline{u - x}}{|u - x|^8}, E_2(v - y) = \frac{1}{w_8}\frac{\overline{v - y}}{|v - y|^8}, \tag{8}$$

Then

$$\begin{cases} D_u E_1(u-x) = E_1(u-x)D_u = 0 \\ D_v E_2(v-y) = E_2(v-y)D_v = 0, \end{cases} \tag{9}$$

Where $w_8 = \dfrac{\pi^4}{3}$ is the area of the unit sphere in $\mathbb{R}^8$.

For later use, we introduce volume element $dv_y = dy_0 \wedge dy_1 \wedge ... \wedge dy_7$ and surface measure

$$d\sigma_y = \sum_{i=0}^{7}(-1)^i e_i d\hat{y}_i = \sum_{i=0}^{7} e_i v_i ds_y$$

$$d\hat{y}_i = dy_0 \wedge ... \wedge dy_{i-1} \wedge dy_{i+1} ... \wedge dy_7 \,, \tag{10}$$

where

$$d\hat{y}_i = dy_0 \wedge ... \wedge dy_{i-1} \wedge dy_{i+1} ... \wedge dy_7$$

stands the removal of dy_i from dv_y for $i = 0,1,...,7$ and v_i is i -th component of unit outward normal and ds_y is the classical Lebesgue surface measure.

Theorem 1 (Stokes theorem 1)[12, 17] Let $U \subset \mathbb{R}^8$ be open and let $S \subset U$ be an7-dimensional compact differentiable oriented manifold with boundary. Then for all $f, g \in C^1(U, O)$,

$$\int_s \left\{ (fD)g + f(Dg) - \sum_{j=0}^{7}\left[\partial_j f, e_j, g\right] \right\} dv_y = \int_{\partial S} f(d\sigma_y g). \tag{11}$$

For convenience, we introduce the Stokes theorem2.

Theorem 2 (Stokes theorem 2) Let $U \subset \mathbb{R}^8$ be open and let $S \subset U$ be an 7-dimensional compact differentiable oriented manifold with boundary. Then for all $f, g \in C^1(U, O)$

$$\int_s \left\{ (fD)g + f(Dg) + \sum_{j=0}^{7}\left[f, e_j, \partial_j g\right] \right\} dv_y = \int_{\partial S} (fd\sigma_y)g. \tag{12}$$

Proof: Applying the divergence theorem, we know that

$$\begin{aligned}
\int_{\partial S} (fd\sigma_y)g &= \int_S \sum_{j=0}^{7} \partial_{y_j}(fd\sigma_y)dy_j g + (fd\sigma_y)\sum_{j=0}^{7}\partial_{y_j} g dy_j \\
&= \int_S \left[\sum_{j=0}^{7} \partial_{y_j}(f\sum_{i=0}^{7}(-1)^i e_i d\hat{y}_i)dy_j g + (f\sum_{i=0}^{7}(-1)^i e_i d\hat{y}_i)\sum_{j=0}^{7}\partial_{y_j} g dy_j \right] \\
&= \int_S \left[\sum_{j=0}^{7} \partial_{y_j}(f\sum_{j=0}^{7}(-1)^j e_j d\hat{y}_j)dy_j g + (f\sum_{j=0}^{7}(-1)^j e_j d\hat{y}_j)\sum_{j=0}^{7}\partial_{y_j} g dy_j \right] \\
&= \int_S \left[(fD)g + \sum_{j=0}^{7}(fe_j)\partial_{y_j} g \right] dv_y \\
&= \int_S \left\{ (fD)g + f(Dg) + \sum_{j=0}^{7}\left[f, e_j, \partial_{y_j} g\right] \right\} dv_y.
\end{aligned} \tag{13}$$

Lemma 1 [16] For any $x, u \in O, x \neq u$,

$$\sum_{j=0}^{7}\left[\partial_{u_j} E_1(u-x), e_j, f(x,u)\right] = 0, x \neq u . \tag{14}$$

Proof: Since

$$E_1(u-x) = \frac{1}{w_8}\frac{\overline{u-x}}{|u-x|^8}$$

and

$$\partial_{u_j}\frac{\overline{u-x}}{|u-x|^8} = \frac{\overline{e_j}}{|u-x|^8} - 8(u_j - x_j)\frac{\overline{u-x}}{|u-x|^{10}}, \tag{15}$$

we get

$$\begin{aligned}
&\sum_{j=0}^{7}\left[\partial_{u_j}\frac{\overline{u-x}}{|u-x|^8}, e_j, f(x,u)\right] \\
&= \sum_{j=0}^{7}\left[\frac{\overline{e_j}}{|u-x|^8} - 8(u_j - x_j)\frac{\overline{u-x}}{|u-x|^{10}}, e_j, f(x,u)\right] \\
&= \sum_{j=0}^{7}\left[\frac{\overline{e_j}}{|u-x|^8}, e_j, f(x,u)\right] - \sum_{j=0}^{7}\left[8(u_j - x_j)\frac{\overline{u-x}}{|u-x|^{10}}, e_j, f(x,u)\right] \\
&= \sum_{j=0}^{7}\frac{1}{|u-x|^8}\left[\overline{e_j}, e_j, f(x,u)\right] - \sum_{j=0}^{7}\frac{8}{|u-x|^{10}}\left[\overline{u-x}, u-x, f(x,u)\right] \\
&= 0,
\end{aligned} \tag{16}$$

by the weak form of associativity (3) in the last step.

Theorem 3 (Cauchy integral formula) Suppose $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^8 \times \mathbb{R}^8$ is nonempty, connected and open set. $\partial\Omega_1, \partial\Omega_2$ are differentiable, oriented and compact Lyapunov surface, if f is biregular function in Ω, then

$$\int_{\partial\Omega_1 \times \partial\Omega_2}\left\{\left[E_1(u-x)(d\sigma_u f(u,v))\right]d\sigma_v\right\}E_2(v-y) = \begin{cases} f(x,y), (x,y) \in \Omega, \\ 0, (x,y) \notin \overline{\Omega}. \end{cases} \tag{17}$$

Proof: When $(x,y) \notin \overline{\Omega}$,

$$\begin{aligned}
&\int_{\partial\Omega_1 \times \partial\Omega_2}\left\{\left[E_1(u-x)(d\sigma_u f(u,v))\right]d\sigma_v\right\}E_2(v-y) \\
&= \int_{\partial\Omega_2}\left\{\left[\int_{\partial\Omega_1} E_1(u-x)(d\sigma_u f(u,v))\right]d\sigma_v\right\}E_2(v-y) \\
&= \int_{\partial\Omega_2}\left\{\int_{\Omega_1}\left[(E_1(u-x)D_u)f(u,v) + E_1(u-x)(D_u f(u,v)) - \sum_{j=0}^{7}\left[\partial u_j E_1(u-x), e_j, f(x,u)\right]\right]dv_u d\sigma_v\right\}E_2(v-y) \\
&= 0.
\end{aligned} \tag{18}$$

The first term of the above proof is due to the nature of kernel function $E_1(u-x)D_u = 0$. Since f is a biregular function $D_u f(u,v) = 0$, then the second term vanishes, and $\left[\partial u_j E_1(u-x), e_j, f(x,u)\right] = 0$ by Lemma 1 in the last term.

When $(x,y) \in \Omega$, we construct a 7-dimensional hyspersphere $B_1(x,\delta) = \left\{u \mid \sum_{i=0}^{7}(u_i - x_i)^2 \leq \delta^2\right\} \subset \Omega_1$,

$B_2(y,\delta)=\left\{v \mid \sum_{i=0}^{7}(v_i-y_i)^2\le\delta^2\right\}\subset\Omega_2$, then

$$\begin{aligned}&\int_{\partial\Omega_1\times\partial\Omega_2}\{[E_1(u-x)(d\sigma_u f(u,v))]d\sigma_v\}E_2(v-y)\\&=\int_{\partial B_1(x,\delta)\times\partial B_2(y,\delta)}\{[E_1(u-x)(d\sigma_u f(u,v))]d\sigma_v\}E_2(v-y)\\&=\theta(\delta).\end{aligned}\tag{19}$$

In the following proof, we will get $\lim_{\delta\to0}\theta(\delta)=f(x,y)$.

$\theta(\delta)=\int_{\partial B_2(y,\delta)}\left\{\left[\int_{\partial B_1(x,\delta)}E_1(u-x)(d\sigma_u f(u,v))\right]d\sigma_v\right\}E_2(v-y)$. Using the Stokes theorem 1, we thus obtain

$$\begin{aligned}&\frac{1}{w_8w_8}\frac{1}{\delta^{16}}\int_{\partial B_2(y,\delta)}\left\{\int_{B_1(x,\delta)}\left[(\overline{u-x}D_u)f(u,v)+\overline{u-x}(D_uf(u,v))-\sum_{j=0}^{7}\left[\partial u_j\overline{u-x},e_j,f(u,v)\right]\right]dv_u d\sigma_v\right\}\overline{v-y}\\&=\frac{1}{w_8w_8}\frac{1}{\delta^{16}}\int_{\partial B_2(y,\delta)}\left\{\int_{B_1(x,\delta)}\left[(\overline{u-x}D_u)f(u,v)+\overline{u-x}(D_uf(u,v))-\sum_{j=0}^{7}\left[\overline{e_j},e_j,f(u,v)\right]\right]dv_u d\sigma_v\right\}\overline{v-y}.\end{aligned}\tag{20}$$

The first term above $\overline{u-x}D_u=8$. The other two terms vanishes, according to definition 2 and the weak form of associativity (3), we get

$$\frac{1}{w_8w_8}\frac{8}{\delta^{16}}\int_{B_1(x,\delta)}\left[\int_{\partial B_2(y,\delta)}(f(u,v)d\sigma_v)\overline{v-y}\right]dv_u.\tag{21}$$

By Stokes theorem 2 in equation (21), we thus obtain

$$\begin{aligned}&\frac{1}{w_8w_8}\frac{8}{\delta^{16}}\int_{B_1(x,\delta)}\left\{\int_{B_2(y,\delta)}\left[(f(u,v)D_v)\overline{v-y}+f(u,v)(D_v\overline{v-y})+\sum_{j=0}^{7}\left[f(u,v),e_j,\partial v_j\overline{v-y}\right]\right]dv_v\right\}dv_u\\&=\frac{1}{w_8w_8}\frac{8}{\delta^{16}}\int_{B_1(x,\delta)}\left\{\int_{B_2(y,\delta)}\left[(f(u,v)D_v)\overline{v-y}+f(u,v)(D_v\overline{v-y})+\sum_{j=0}^{7}\left[f(u,v),e_j,\overline{e_j}\right]\right]dv_v\right\}dv_u\\&=\frac{1}{w_8w_8}\frac{64}{\delta^{16}}\int_{B_1(x,\delta)}\left[\int_{B_2(y,\delta)}f(u,v)dv_v\right]dv_u.\end{aligned}\tag{22}$$

The proof is similar to (20).

Therefore, for sufficiently small δ, we have $f(u,v)=f(u,y)+\theta(\delta), f(u,y)=f(x,y)+\theta(\delta)$, then

$$\begin{aligned}&\lim_{\delta\to0}\theta(\delta)=\frac{1}{w_8w_8}\frac{64}{\delta^8}\int_{B_1(x,\delta)}\left[\int_{B_2(y,\delta)}(f(u,y)+\theta(\delta))dv_v\right]dv_u\\&=\frac{8}{\delta^8}\int_{B_1(x,\delta)}f(u,y)dv_u\\&=\frac{8}{\delta^8}\int_{B_1(x,\delta)}(f(x,y)+\theta(\delta))dv_u\\&=f(x,y).\end{aligned}\tag{23}$$

Then the proof of the result is completed.

Corollary: Let $\Omega=\Omega_1\times\Omega_2$ be as stated above, $\partial\Omega_1,\partial\Omega_2$ are differentiable, oriented and compact Lyapunov surface, if f is biregular function in Ω , then we have

$$\int_{\partial\Omega_1\times\partial\Omega_2}\{[E_1(u-x)(d\sigma_u)]d\sigma_v\}E_2(v-y)=\begin{cases}1, for(x,y)\in\Omega,\\0, for(x,y)\notin\overline{\Omega}.\end{cases}\tag{24}$$

Theorem 4 (Mean value theorem) Let $B(x,\delta)\subset\mathbb{R}^8, B(y,\delta)\subset\mathbb{R}^8$ and f is a biregular function in $B(x,\delta_1)\times B(y,\delta_2)$, we have

$$f(x,y)=\frac{1}{\delta_1^8 v_8\delta_2^8 v_8}\int_{B(x,\delta_1)\times B(y,\delta_2)}f(u,y)dv_u dv_v.\tag{25}$$

Proof: Applying Cauchy integral formula, we know
$f(x,y)=\frac{1}{\delta_1{}^8 w_8\delta_2{}^8 w_8}\int_{\partial B(x,\delta_1)\times\partial B(y,\delta_2)}\{[\overline{u-x}(d\sigma_u f(u,v))]d\sigma_v\}\overline{v-y}$. Let's use Stokes theorem 1, we have

$$\begin{aligned}&\frac{1}{\delta_1{}^8 w_8\delta_2{}^8 w_8}\int_{\partial B(y,\delta_2)}\left\{\int_{B(x,\delta_1)}\left[(\overline{u-x}D_u)f(u,v)+\overline{u-x}(D_u f(u,v))-\sum_{j=0}^{7}\left[\partial u_j\overline{u-x},e_j,f(u,v)\right]\right]dv_u d\sigma_v\right\}\overline{v-y}\\&=\frac{1}{\delta_1{}^8 w_8\delta_2{}^8 w_8}\int_{\partial B(y,\delta_2)}\left\{\int_{B(x,\delta_1)}\left[(\overline{u-x}D_u)f(u,v)+\overline{u-x}(D_u f(u,v))-\sum_{j=0}^{7}\left[\overline{e_j},e_j,f(u,v)\right]\right]dv_u d\sigma_v\right\}\overline{v-y},\end{aligned}\tag{26}$$

By the weak form of associativity (3) in the last term, and the other two terms are due to
$D_u f(u,v)=0, f(u,v)D_v=0;\overline{u-x}D_u=8, D_v\overline{v-y}=8,$ we get

$$\frac{1}{\delta_1{}^8 w_8\delta_2{}^8 w_8}\int_{\partial B(y,\delta_2)}\left\{\int_{B(x,\delta_1)}[8f(u,v)]dv_u d\sigma_v\right\}\overline{v-y}=\frac{8}{\delta_1{}^8 w_8\delta_2{}^8 w_8}\int_{B(x,\delta_1)}\left\{\int_{\partial B(y,\delta_2)}(f(u,v)d\sigma_v)\overline{v-y}\right\}dv_u.\tag{27}$$

By Stokes theorem 2, we have

$$\begin{aligned}&\frac{8}{\delta_1{}^8 w_8\delta_2{}^8 w_8}\int_{B(x,\delta_1)\times B(y,\delta_2)}\left\{(f(u,v)D_v)\overline{v-y}+f(u,v)(D_v\overline{v-y})+\sum_{j=0}^{7}\left[f(u,v),e_j,\partial v_j\overline{v-y}\right]\right\}dv_u dv_v\\&=\frac{8}{\delta_1{}^8 w_8\delta_2{}^8 w_8}\int_{B(x,\delta_1)\times B(y,\delta_2)}\left\{(f(u,v)D_v)\overline{v-y}+f(u,v)(D_v\overline{v-y})+\sum_{j=0}^{7}\left[f(u,v),e_j,\overline{e_j}\right]\right\}dv_u dv_v\\&=\frac{64}{\delta_1{}^8 w_8\delta_2{}^8 w_8}\int_{B(x,\delta_1)\times B(y,\delta_2)}f(u,v)dv_u dv_v\\&=\frac{1}{\delta_1{}^8 w_8\delta_2{}^8 w_8}\int_{B(x,\delta_1)\times B(y,\delta_2)}f(u,v)dv_u dv_v.\end{aligned}\tag{28}$$

The proof is similar to the above manner. Hence $v_8=\frac{w_8}{8}$ is volume of the 7-dimensional unit ball.

5. Conclusion

In this paper, using the methods in dealing with the Cauchy integral formula of biregular function in Clifford analysis and regular function in octonionic analysis, we obtain the Cauchy integral formula and mean value theorem for biregular function in octonionic analysis.

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