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An Analytic Proof of Some Part of Keith-Xiong's Theorem

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Abstract: In the theory of partitions, Euler's partition theorem involving odd parts and different parts is one of the famous theorems. It states that the number of partitions of an integer n into odd parts is equal to the number of partitions of n into different parts. By intepretting odd parts as parts congruent to 1 modulo 2, the second author and Keith provided a completely generalization about Euler's partition theorem involving odd parts and different parts for all moduli and provide new companions to Rogers-Ramanujan-Andrews-Gordon identities related to this theorem. They gave a combinatorial proof of the theorem by establishing bijection. In this note, we will offer an anclytic view point of this beautiful theorem. We use q-series and generating function theories to provide an analytic style proof for some cases of Keith-Xiong's theorem. By defining basic units and special units, the basic units in the partitions are divided into two categories, and then the number between the basic units in the special units is classified, and all the cases when $m = 3$ and alternative sum type $(\Sigma, 2)$ are given, our method is verifying the generating functions of both sides satisfying the same recurrences.

Keywords: Partition, $q-$series, Generating Function

1. Introduction

Euler proved that the number of partitions of n into odd parts is equal to the number of partitions n into different parts. Over the years, there have been a lot of refinements and generalizations around it [1, 2, 3, 4, 5, 6, 7, 8]. By intepretting odd parts as parts congruent to 1 modulo 2, the second author and Keith provided a completely generalization [9]. Fix a natural number $m \geq 2$, a partition with all parts not congruent to zero modulo m is said to a partition with length type $(l_1, l_2, l_3, \ldots, l_{m-2}, l_{m-1})$, where for $1 \leq i \leq m-1$, l_i is the number of parts of the partition which are congruent to i modulo m. A m-regular partition is said to be a partition with alternative sum type $(\Sigma_1, \Sigma_2, \ldots, \Sigma_{m-2}, \Sigma_{m-1})$, if we write the partition as the form $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_{km}$ $(k \geq 1)$ by allowing zero as a part, then $\Sigma_i = (\lambda_i - \lambda_{i+1}) + (\lambda_{(m+i)} - \lambda_{(m+i+1)}) + (\lambda_{(2m+i)} - \lambda_{(2m+i+1)}) + \cdots + (\lambda_{(km-m+i)} - \lambda_{(km-m+i+1)})$. The main result in [9] is the following theorem.

Theorem 1.1 Let $m \geq 2$, let P be the set of partitions with each part can be repeated at most $m-1$ times, this implies their alternating sum types can not be $(0, 0, \ldots, 0)$. Let Q be the set of partitions with no parts $\equiv 0 \pmod m$. Then we have the partition identity:

$$\sum_{\lambda \in P} z_1^{\Sigma_1(\lambda)} z_2^{\Sigma_2(\lambda)} \cdots z_{m-1}^{\Sigma_{m-1}(\lambda)} q^{|\lambda|} = \sum_{\mu \in Q} z_1^{l_1(\mu)} z_2^{l_2(\mu)} \cdots z_{m-1}^{l_{m-1}(\mu)} q^{|\mu|}.$$

For example, take $n = 7, m = 3$, the partitions of 7 with alternative sum type $(2, 1)$ are $4 + 2 + 1$ and $3 + 2 + 1 + 1$. The partitions of 7 with length type $(2, 1)$ are $5 + 1 + 1$ and $4 + 2 + 1$. In order to prove this theorem, they construct a beautiful bijection [9, 11, 12, 13, 14, 15].In this note, we use q-series and generating function theories to provide an analytic style proof for $m = 3$ and alternative sum type $(\Sigma, 2)$. We hope to offer an anclytic view point of this beautiful theorem. Our method is verifying the generating functions of both sides satisfying the same recurrences.

2. Analytic Proof for $m = 3$ and Alternative Sum Type $(\Sigma, 2)$

We begin with the following lemma.

Lemma 2.1 Let $m \geq 2$, $1 \leq i \leq m-1$. Let Q_i be the set of partitions with all parts are congruent to i modulo m, let $f_{Q_i}(z,q)$ be the generating function for Q_i, $b_i(l;n)$ be the number of partitions of n with l parts and all parts $\equiv i \pmod m$. Let $f_Q(z_1, z_2, \dots, z_{m-1}, q)$ be the generating function of Q, and $b(l_1, l_2, \dots, l_{m-1}; n)$ be the number of partitions of n with length type $(l_1, l_2, \dots, l_{m-1})$. That is

$$f_{Q_i}(z,q) = \sum_{\lambda \in Q_i} z^{l(\lambda)} q^{|\lambda|} = \sum_{l \geq 0, n \geq 0} b_i(l;n) z^l q^n, \quad l(\lambda) = \text{ length of } \lambda,$$

and

$$f_Q(z_1, z_2, \dots, z_{m-1}, q) = \sum_{\lambda \in Q} z_1^{l_1(\lambda)} z_2^{l_2(\lambda)} \cdots z_{m-1}^{l_{m-1}(\lambda)} q^{|\lambda|} = \sum_{l_i, n \geq 0} b(l_1, l_2, \dots, l_{m-1}; n) z_1^{l_1} z_2^{l_2} \cdots z_{m-1}^{l_{m-1}} q^n.$$

then the generating function of Q satisfies

$$f_Q(z_1, z_2, \dots, z_{m-1}, q) = \prod_{i=1}^{m-1} f_{Q_i}(z_i, q).$$

equivalently,

$$b(l_1, l_2, \dots, l_{m-1}; n) = \sum_{n_i \geq 0, n_1 + \cdots + n_{m-1} = n} b_1(l_1; n_1) b_2(l_2; n_2) \cdots b_{m-1}(l_{m-1}; n_{m-1}).$$

Proof. This is a simple application of the fundamental counting principle. In order to construct a partition of n with length type $(l_1, l_2, \dots, l_{m-1})$, we firstly decomposition n into $m-1$ positive integers $n_1, n_2, \dots, n_{m-1}$, for each n_i, choose a partition of n_i with l_i parts and all parts $\equiv i \pmod m$. Put all parts of these m-1 partitions together, we get a partition of n with length type $(l_1, l_2, \dots, l_{m-1})$. When considering all partitions of n_i with the given conditions and all possible decompositions of n, we get all partitions of n with given the length type $(l_1, l_2, \dots, l_{m-1})$.

From this lemma and the facts that the function

$$\frac{q^{2i}}{(1-q^3)(1-q^6)} \quad (i = 1, 2)$$

generates all partitions with exactly two parts and each part $\equiv i \pmod 3$ and $\frac{q^j}{1-q^m}(1 \leq j \leq m-1)$ generates all partitions with only one part and this part is congruent to j modulo m, we get the following results:

$$\sum_{l,n \geq 0} b(l,2;n) z^l q^n = \left(\sum_{l,n \geq 0} b(l,0;n) z^l q^n \right) \frac{q^4}{(1-q^3)(1-q^6)}, \tag{1}$$

$$\sum_{l,n \geq 0} b(2,l;n) z^l q^n = \left(\sum_{l,n \geq 0} b(0,l;n) z^l q^n \right) \frac{q^2}{(1-q^3)(1-q^6)}, \tag{2}$$

$$\sum_{l,n \geq 0} \tilde{b}_{i;j}(l,1;n) z^l q^n = \left(\sum_{l,n \geq 0} b(0, \dots, 0, l, 0, \dots, 0; n) z^l q^n \right) \frac{q^j}{1-q^m}. \tag{3}$$

Where $1 \leq j \leq m-1, j \neq i$ and $\tilde{b}_{i;j}(l,1;n) = b(0,0,\dots,0,l,0,\dots,0,1,0,\dots,0;n)$, which is the number of partitions of n with exactly l parts $\equiv i \pmod m$ and one part $\equiv j \pmod m$.

Now we prove Theorem 1.1 for the type $(\Sigma, 2)$. Let $a(\Sigma_1, \Sigma_2; n)$ denote the number of partitions of n with the alternating sum type (Σ_1, Σ_2). Theorem 1.1 is equivalent to the claim: for any $\Sigma \geq 0$, $a(\Sigma, 2; n) = b(\Sigma, 2; n)$. Since $a(\Sigma, 0; n) = b(\Sigma, 0; n)$, this is due to Pak-Postnikov[7, 9]. So it is only to prove the generating function of $a(\Sigma, 2; n)$ satisfies the following identity by comparing with the identity (1):

$$\sum_{\Sigma, n \geq 0} a(\Sigma, 2; n) z^\Sigma q^n = \left(\sum_{\Sigma, n \geq 0} a(\Sigma, 0; n) z^\Sigma q^n \right) \frac{q^4}{(1-q^3)(1-q^6)}. \tag{4}$$

Let λ be a partition with parts repeated times ≤ 2 and the alternating sum type $(\Sigma, 2)$. The basic units in λ have two cases:

Case A. The basic units consist in forms:

$$\lambda_1 > \lambda_2 = \lambda_3 \quad \text{and two forms} \quad \lambda'_1 \geq \lambda'_2 \stackrel{1}{>} \lambda'_3.$$

Where $\lambda'_2 \stackrel{1}{>} \lambda'_3$ means $\lambda'_2 - \lambda'_3 = 1$. Here is an example with four basic units:

$$10 > 8 = 8 > 7 = 7 > 6 = 6 > 5 = 5 > 2 > 2 > 1,$$

the first unit and third unit are the form $\lambda_1 > \lambda_2 = \lambda_3$, the second unit and fourth unit (with black colour) are the form $\lambda'_1 \geq \lambda'_2 \stackrel{1}{>} \lambda'_3$.

Case B. The basic units consist in forms:

$$\lambda_1 > \lambda_2 = \lambda_3 \quad \text{and one form} \quad \lambda'_1 \geq \lambda'_2 \stackrel{2}{>} \lambda'_3.$$

Where $\lambda'_2 \stackrel{2}{>} \lambda'_3$ means $\lambda'_2 - \lambda'_3 = 2$. We give an example for this case, $8 > 7 = 7 > 6 > 4 = 4 > 3 > 2$, where the first unit is $\lambda_1 = 8, \lambda_2 = \lambda_3 = 7$, the second unit is $\lambda_4 = 6, \lambda_5 = \lambda_6 = 4$, and the third unit is $\lambda_7 = 3, \lambda_8 = 2, \lambda_9 = 0$.

We call the types $\lambda'_1 \geq \lambda'_2 \stackrel{2}{>} \lambda'_3$ and $\lambda'_1 \geq \lambda'_2 \stackrel{1}{>} \lambda'_3$ to be special units.

We firstly consider *Case A*.

Lemma 2.2 The generating function for the number of partitions of fixed length $3n+3$ with alternating sum type $(\Sigma, 2)$ and belonging to the *Case A* is the sum of the following n terms, where $n \geq 1$.

$$\frac{z^{n+1} q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}} \cdot \left(\sum_{k=1}^{n-1} \left(\frac{1}{z^2 q^{3k+1}} + \frac{1-q^{3k}}{zq^{3k}} \right) (n \geq 2) + \left(\frac{q^2}{z^2} + \frac{q^3(1-q^{3n})}{z} \right) (n \geq 1) \right), \tag{5}$$

$$\frac{z^{n+1} q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}} \left(\sum_{k=1}^{n-2} \frac{1}{z^2 q^{6k+4}} (n \geq 3) + \frac{1}{z^2 q^{3n-5}} (n \geq 2) \right), \tag{6}$$

$$\frac{z^{n+1} q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}} \left(\sum_{k=1}^{n-3} \frac{1}{z^2 q^{6k+7}} (n \geq 4) + \frac{1}{z^2 q^{3n-8}} (n \geq 3) \right), \tag{7}$$

$$\cdots$$

$$\frac{z^{n+1} q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}} \left(\sum_{k=1}^{n-(n-1)} \frac{1}{z^2 q^{6k+3n-5}} + \frac{1}{z^2 q^4} \right), \tag{8}$$

$$\frac{z^{n+1} q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}} \frac{1}{z^2 q}. \tag{9}$$

Here we write them as a list so that we can easily see how each term of them corresponds to the following analysis and the inequalities on n in brackets means the corresponding terms requiring n satisfying this condition. Moreover we use the standard q-Pochhammer symbol notations for simplicity defined by

$$(a;q)_n := \prod_{i=0}^{n-1} (1 - aq^i) \text{ and } (a;q)_\infty := \lim_{L \to \infty} (a;q)_n.$$

But in the proof of our main results, we sometimes choose product forms instead of q-Pochhammer symbol so that readers can easily track our transformations.

Lemma 2.3 The generating function for the number of partitions of fixed length $3n+2$ with the alternating sum type $(\Sigma, 2)$ and belonging to the *Case A* is the sum of the following n terms.

$$\frac{z^{n+1} q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_n} \left(\frac{q^2}{z^2} + \frac{q^3(1-q^{3n})}{z} \right) (n \geq 1), \tag{10}$$

$$\frac{z^{n+1} q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_n} \frac{1}{z^2 q^{3n-5}} (n \geq 2), \tag{11}$$

$$\frac{z^{n+1} q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_n} \frac{1}{z^2 q^{3n-8}} (n \geq 3), \tag{12}$$

$$\cdots$$

$$\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_n}\frac{1}{z^2q^4}, \tag{13}$$

$$\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_n}\frac{1}{z^2q}. \tag{14}$$

Lemma 2.4 The generating function for the number of partitions of fixed length $3n+1$ with alternating sum type $(\Sigma, 2)$ and belonging to the *Case A* is the sum of the following $n-1$ terms.

$$\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_n}\sum_{k=1}^{n-1}\left(\frac{1}{z^2q^{3k+1}}+\frac{1-q^{3k}}{zq^{3k}}\right)(n\geq 2), \tag{15}$$

$$\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_n}\sum_{k=1}^{n-2}\frac{1}{z^2q^{6k+4}}(n\geq 3), \tag{16}$$

$$\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_n}\sum_{k=1}^{n-3}\frac{1}{z^2q^{6k+7}}(n\geq 4), \tag{17}$$

$$\cdots$$

$$\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_n}\sum_{k=1}^{n-(n-1)}\frac{1}{z^2q^{6k+3n-5}}. \tag{18}$$

Proof of Lemma 2.2. Let λ be a partition of length $3n+3$ and its basic units belonging to the *Case A*. Note that λ has $n+1$ basic units, two of them are special units. We classify such partitions into n classes by the distance of two special units. Here the distance 0 means that there is no basic unit of the type $\lambda_1 > \lambda_2 = \lambda_3$ between the two special units. The distance 1 means there is only one basic unit of type $\lambda_1 > \lambda_2 = \lambda_3$ between the two special units, etc. For example, the partition

$$9 > 8 = 8 > 7 = 7 > 6 > 5 > 4 = 4 > 3 > 2 > 1,$$

has the distance between special units (the color is black) 1. By a partition of class d, $0 \leq d \leq n-1$, it means the distance between special units is d. We consider each class as follows.

Distance 0. Let λ be such a partition, if the two special units are not the last two basic units, then it has the form:

$$\lambda_1 > \lambda_2 = \lambda_3 > \cdots > \lambda_{3k-5} > \lambda_{3k-4} = \lambda_{3k-3} > \lambda_{3k-2} \geq \lambda_{3k-1} \overset{1}{>} \lambda_{3k} >$$
$$\lambda_{3k+1} \geq \lambda_{3k+2} \overset{1}{>} \lambda_{3k+3} \geq \lambda_{3k+4} > \lambda_{3k+5} = \lambda_{3k+6} > \cdots > \lambda_{3n+1} > \lambda_{3n+2} = \lambda_{3n+3},$$

or the form:

$$\lambda_1 > \lambda_2 = \lambda_3 > \cdots > \lambda_{3k-5} > \lambda_{3k-4} = \lambda_{3k-3} > \lambda_{3k-2} \geq \lambda_{3k-1} \overset{1}{>} \lambda_{3k} =$$
$$\lambda_{3k+1} > \lambda_{3k+2} \overset{1}{>} \lambda_{3k+3} \geq \lambda_{3k+4} > \lambda_{3k+5} = \lambda_{3k+6} > \cdots > \lambda_{3n+1} > \lambda_{3n+2} = \lambda_{3n+3}.$$

Where $\lambda_{3n+3} > 0$ and $1 \leq k \leq n-1$. By considering conjugates of such partitions, the standard partition analysis gives the generating function for the first case is

$$\begin{aligned}
&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{zq^{3k-5}}{1-zq^{3k-5}}\frac{q^{3k-3}}{1-q^{3k-3}}\frac{1}{1-zq^{3k-2}}\cdot q^{3k-1}\cdot\frac{q^{3k}}{1-q^{3k}}\\
&\cdot\frac{1}{1-zq^{3k+1}}\cdot q^{3k+2}\frac{1}{1-q^{3k+3}}\cdot\frac{zq^{3k+4}}{1-zq^{3k+4}}\cdots\frac{zq^{3n+1}}{1-zq^{3n+1}}\frac{q^{3n+3}}{1-q^{3n+3}}\\
=&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{zq^{3n+1}}{1-zq^{3n+1}}\frac{q^{3n+3}}{1-q^{3n+3}}\cdot\frac{q^2}{z^2}\cdot\frac{1}{q^{3k+3}}\\
=&\frac{z^{n+1}q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}}\frac{1}{z^2q^{3k+1}},
\end{aligned} \tag{19}$$

and the generating function for the second case is

$$\begin{aligned}
&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{zq^{3k-5}}{1-zq^{3k-5}}\frac{q^{3k-3}}{1-q^{3k-3}}\frac{1}{1-zq^{3k-2}}\cdot q^{3k-1}\\
&\cdot\frac{zq^{3k+1}}{1-zq^{3k+1}}\cdot q^{3k+2}\frac{1}{1-q^{3k+3}}\cdot\frac{zq^{3k+4}}{1-zq^{3k+4}}\cdots\frac{zq^{3n+1}}{1-zq^{3n+1}}\frac{q^{3n+3}}{1-q^{3n+3}}\\
=&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{zq^{3n+1}}{1-zq^{3n+1}}\frac{q^{3n+3}}{1-q^{3n+3}}\cdot\frac{q^3(1-q^{3k})}{z}\cdot\frac{1}{q^{3k+3}}\\
=&\frac{z^{n+1}q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}}\frac{1-q^{3k}}{zq^{3k}}.
\end{aligned}\tag{20}$$

If the special units are the last two basic units, then it has the form:

$$\lambda_1>\lambda_2=\lambda_3>\cdots>\lambda_{3n-5}>\lambda_{3n-4}=\lambda_{3n-3}>\lambda_{3n-2}\geq\lambda_{3n-1}\overset{1}{>}\lambda_{3n}>\lambda_{3n+1}\geq\lambda_{3n+2}\overset{1}{>}\lambda_{3n+3},(\lambda_{3n+3}>0),$$

or the form:

$$\lambda_1>\lambda_2=\lambda_3>\cdots>\lambda_{3n-5}>\lambda_{3n-4}=\lambda_{3n-3}>\lambda_{3n-2}\geq\lambda_{3n-1}\overset{1}{>}\lambda_{3n}=\lambda_{3n+1}>\lambda_{3n+2}\overset{1}{>}\lambda_{3n+3},(\lambda_{3n+3}>0).$$

By considering conjugates of such partitions, the standard partition analysis gives the generating function for the first case is

$$\begin{aligned}
&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{zq^{3n-5}}{1-zq^{3n-5}}\frac{q^{3n-3}}{1-q^{3n-3}}\frac{1}{1-zq^{3n-2}}\cdot q^{3n-1}\cdot\frac{q^{3n}}{1-q^{3n}}\cdot\frac{1}{1-zq^{3n+1}}\cdot q^{3n+2}\frac{q^{3n+3}}{1-q^{3n+3}}\\
=&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{zq^{3n+1}}{1-zq^{3n+1}}\frac{q^{3n+3}}{1-q^{3n+3}}\cdot\frac{q^2}{z^2}\\
=&\frac{z^{n+1}q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}}\frac{q^2}{z^2}.
\end{aligned}\tag{21}$$

And the generating function for the second case is

$$\begin{aligned}
&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{zq^{3n-5}}{1-zq^{3n-5}}\frac{q^{3n-3}}{1-q^{3n-3}}\frac{1}{1-zq^{3n-2}}\cdot q^{3n-1}\cdot\frac{zq^{3n+1}}{1-zq^{3n+1}}\cdot q^{3n+2}\frac{q^{3n+3}}{1-q^{3n+3}}\\
=&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{zq^{3n+1}}{1-zq^{3n+1}}\frac{q^{3n+3}}{1-q^{3n+3}}\cdot\frac{q^3(1-q^{3n})}{z}\\
=&\frac{z^{n+1}q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}}\frac{q^3(1-q^{3n})}{z}.
\end{aligned}\tag{22}$$

Add up (19) and (20), summing k from 1 to $n-1$ and then add them to (21) and (22), the sum is

$$\frac{z^{n+1}q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}}\left(\sum_{k=1}^{n-1}\left(\frac{1}{z^2q^{3k+1}}+\frac{1-q^{3k}}{zq^{3k}}\right)+\left(\frac{q^2}{z^2}+\frac{q^3(1-q^{3n})}{z}\right)\right),$$

Which is the term (5) in Lemma 2.2. When special units are not the last basic unit, such a partition has at least three basic units, hence in the term

$$\sum_{k=1}^{n-1}\left(\frac{1}{z^2q^{3k+1}}+\frac{1-q^{3k}}{zq^{3k}}\right),$$

n should satisfy $n\geq 2$. The term

$$\frac{q^2}{z^2}+\frac{q^3(1-q^{3n})}{z}$$

corresponds to the special units being the last two basic units, it implies λ has at least two basic units, hence $n\geq 1$. In the following cases, we have similar analysis on inequalities involving n.

Distance 1. In this case, the two special units must appear in the following form:

$$\lambda_{3k-2} \geq \lambda_{3k-1} \overset{1}{>} \lambda_{3k} \geq \lambda_{3k+1} > \lambda_{3k+2} = \lambda_{3k+3} > \lambda_{3k+4} \geq \lambda_{3k+5} \overset{1}{>} \lambda_{3k+6},$$

if any special unit does not appear as the last basic unit, hence k can take from 1 to $n-2$, or the form:

$$\lambda_{3n-5} \geq \lambda_{3n-4} \overset{1}{>} \lambda_{3n-3} \geq \lambda_{3n-2} > \lambda_{3n-1} = \lambda_{3n} > \lambda_{3n+1} \geq \lambda_{3n+2} \overset{1}{>} \lambda_{3n+3},$$

if one special unit is exactly the last basic unit. The generating function for the former is

$$\begin{aligned}
&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{zq^{3k-5}}{1-zq^{3k-5}}\frac{q^{3k-3}}{1-q^{3k-3}}\frac{1}{1-zq^{3k-2}}\cdot q^{3k-1}\cdot\frac{1}{1-q^{3k}}\cdot\frac{zq^{3k+1}}{1-zq^{3k+1}}\cdot\frac{q^{3k+3}}{1-q^{3k+3}}\\
&\cdot\frac{1}{1-zq^{3k+4}}\cdot q^{3k+5}\cdot\frac{1}{1-q^{3k+6}}\cdot\frac{zq^{3k+7}}{1-zq^{3k+7}}\cdots\frac{zq^{3n+1}}{1-zq^{3n+1}}\frac{q^{3n+3}}{1-q^{3n+3}}\\
=&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{zq^{3n+1}}{1-zq^{3n+1}}\frac{q^{3n+3}}{1-q^{3n+3}}\cdot\frac{1}{z^2q^{6k+4}}\\
=&\frac{z^{n+1}q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}}\frac{1}{z^2q^{6k+4}}, (n\geq 3),
\end{aligned}\tag{23}$$

and for the later is

$$\begin{aligned}
&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{zq^{3n-8}}{1-zq^{3n-8}}\frac{q^{3n-6}}{1-q^{3n-6}}\frac{1}{1-zq^{3n-5}}\cdot q^{3n-4}\cdot\frac{1}{1-q^{3n-3}}\cdot\\
&\frac{zq^{3n-2}}{1-zq^{3n-2}}\cdot\frac{q^{3n}}{1-q^{3n}}\cdot\frac{1}{1-zq^{3n+1}}\cdot q^{3n+2}\cdot\frac{q^{3n+3}}{1-q^{3n+3}}\\
=&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{zq^{3n+1}}{1-zq^{3n+1}}\frac{q^{3n+3}}{1-q^{3n+3}}\cdot\frac{1}{z^2q^{3n-5}}\\
=&\frac{z^{n+1}q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}}\frac{1}{z^2q^{3n-5}}, (n\geq 2).
\end{aligned}\tag{24}$$

Sum (23) for k from 1 to $n-2$ and then add it to (24), we get the term

$$\frac{z^{n+1}q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}}\left(\sum_{k=1}^{n-2}\frac{1}{z^2q^{6k+4}}(n\geq 3)+\frac{1}{z^2q^{3n-5}}(n\geq 2)\right),$$

which is the term (6) in Lemma 2.2.

Distance d satisfies $2 \leq d \leq n-2$. The two special units must appear in the form:

$$\lambda_{3k-2} \geq \lambda_{3k-1} \overset{1}{>} \lambda_{3k} \geq \lambda_{3k+1} > \lambda_{3k+2} = \lambda_{3k+3} > \cdots = \lambda_{3k+3d} > \lambda_{3k+3d+1} \geq \lambda_{3k+3d+2} \overset{1}{>} \lambda_{3k+3d+3},$$

corresponding to the last special unit being not the last basic unit, hence k can take from 1 to $n-d-1$ for fixed d, or the form:

$$\lambda_{3n-3d-2} \geq \lambda_{3n-3d-1} \overset{1}{>} \lambda_{3n-3d} \geq \lambda_{3n-3d+1} > \cdots > \lambda_{3n+1} \geq \lambda_{3n+2} \overset{1}{>} \lambda_{3n+3},$$

corresponding to one special unit exactly being the last basic unit. The generating function for the former is

$$\begin{aligned}
&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{zq^{3k-5}}{1-zq^{3k-5}}\frac{q^{3k-3}}{1-q^{3k-3}}\frac{1}{1-zq^{3k-2}}\cdot q^{3k-1}\cdot\frac{1}{1-q^{3k}}\cdot\frac{zq^{3k+1}}{1-zq^{3k+1}}\cdots\frac{zq^{3k+3d-2}}{1-zq^{3k+3d-2}}\\
&\cdot\frac{q^{3k+3d}}{1-q^{3k+3d}}\frac{1}{1-zq^{3k+3d+1}}\cdot q^{3k+3d+2}\cdot\frac{1}{1-q^{3k+3d+3}}\cdot\frac{zq^{3k+3d+4}}{1-zq^{3k+3d+4}}\cdots\frac{zq^{3n+1}}{1-zq^{3n+1}}\frac{q^{3n+3}}{1-q^{3n+3}}\\
=&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{zq^{3n+1}}{1-zq^{3n+1}}\frac{q^{3n+3}}{1-q^{3n+3}}\cdot\frac{1}{z^2q^{6k+3d+1}}\\
=&\frac{z^{n+1}q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}}\frac{1}{z^2q^{6k+3d+1}}, (2\leq d\leq n-2, 1\leq k\leq n-d-1).
\end{aligned}\tag{25}$$

And for the later is

$$\begin{aligned}
&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n-3d-3}}{1-q^{3n-3d-3}}\frac{1}{1-zq^{3n-3d-2}}\cdot q^{3n-3d-1}\cdot\frac{1}{1-q^{3n-3d}}\cdot\\
&\frac{zq^{3n-3d+1}}{1-zq^{3n-3d+1}}\cdots\frac{q^{3n}}{1-q^{3n}}\cdot\frac{1}{1-zq^{3n+1}}\cdot q^{3n+2}\cdot\frac{q^{3n+3}}{1-q^{3n+3}}\\
=&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{zq^{3n+1}}{1-zq^{3n+1}}\frac{q^{3n+3}}{1-q^{3n+3}}\cdot\frac{1}{z^2q^{3n-3d-2}}\\
=&\frac{z^{n+1}q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}}\frac{1}{z^2q^{3n-3d-2}},(2\le d\le n-1).
\end{aligned}\tag{26}$$

For each $2\le d\le n-2$, sum (25) for k from 1 to $n-d-1$ and then add up the term in (26) corresponding to d, we get the term (d) in Lemma 2.2.

Distance $n-1$. This is the last case of Lemma 2.2, it corresponds to the distance $n-2$ being maximal. Hence one special unit is the first basic unit and another special unit is the last basic unit. λ has the form:

$$\lambda_1\ge\lambda_2\overset{1}{>}\lambda_3\ge\lambda_4>\cdots>\lambda_{3n+1}\ge\lambda_{3n+2}\overset{1}{>}\lambda_{3n+3}.$$

The generating function is

$$\begin{aligned}
&\frac{1}{1-zq}\cdot q^2\frac{1}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{q^{3n}}{1-q^{3n}}\frac{1}{1-zq^{3n+1}}\cdot q^{3n+2}\cdot\frac{q^{3n+3}}{1-q^{3n+3}}\\
=&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{zq^{3n+1}}{1-zq^{3n+1}}\frac{q^{3n+3}}{1-q^{3n+3}}\cdot\frac{1}{z^2q}\\
=&\frac{z^{n+1}q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}}\frac{1}{z^2q}.
\end{aligned}$$

Which is the term (9) in Lemma 2.2. We complete the proof of Lemma 2.2.

Proof of Lemma 2.3. The analysis above holds for the case of λ has length $3n+2$, except that in each case ($0\le d\le n-1$, d = distance), one special unit must be the last unit and $\lambda_{3n+2}=1$.

We use the case of distance 0 to illustrate them. λ has the form:

$$\lambda_1>\lambda_2=\lambda_3>\cdots>\lambda_{3n-2}\ge\lambda_{3n-1}\overset{1}{>}\lambda_{3n}>\lambda_{3n+1}\ge\lambda_{3n+2},(\lambda_{3n+2}=1),$$

or the form:

$$\lambda_1>\lambda_2=\lambda_3>\cdots>\lambda_{3n-2}\ge\lambda_{3n-1}\overset{1}{>}\lambda_{3n}=\lambda_{3n+1}>\lambda_{3n+2},(\lambda_{3n+2}=1).$$

The generating function for the former is

$$\begin{aligned}
&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n-3}}{1-q^{3n-3}}\frac{1}{1-zq^{3n-2}}q^{3n-1}\frac{q^{3n}}{1-q^{3n}}\frac{1}{1-zq^{3n+1}}q^{3n+2}\\
=&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{zq^{3n+1}}{1-zq^{3n+1}}\frac{q^2}{z^2}\\
=&\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_n}\frac{q^2}{z^2}.
\end{aligned}\tag{27}$$

The generating function for the later is

$$\begin{aligned}
&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n-3}}{1-q^{3n-3}}\frac{1}{1-zq^{3n-2}}q^{3n-1}\frac{zq^{3n+1}}{1-zq^{3n+1}}q^{3n+2}\\
=&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{zq^{3n+1}}{1-zq^{3n+1}}\frac{q^3(1-q^{3n})}{z}\\
=&\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_n}\frac{q^3(1-q^{3n})}{z}.
\end{aligned}\tag{28}$$

Add up (27) and (28), we get the term (10) in Lemma 2.3. Such a partition has at least two basic units, hence $n \geq 1$. By the similar analysis in Lemma 2.2, we find that for the cases of distance from 1 to $n-1$, partitions with length $3n+2$ and one special unit is the last unit can be obtained from partitions with length $3n+3$ and one special unit is the last basic unit by omitting the part λ_{3n+3} and letting $\lambda_{3n+2} = 1$. Therefore, in each case, the generating function for partitions with length $3n+2$ and one special unit being the last basic unit is the same as that of the partitions of length $3n+3$ with one special unit being the last basic unit except without the factor

$$\frac{q^{3n+3}}{1-q^{3n+3}}.$$

They correspond to the terms (11) to (14) in Lemma 2.3.

We complete the proof of Lemma 2.3.

Proof of Lemma 2.4. Since such a partition has length $3n+1$, hence the last basic unit can not be a special unit. Therefore, the range of distance between two special units is from 1 to $n-2$. As the analysis of the partitions with length $3n+3$, we find in each case (distance from 0 to $n-2$), partitions with length $3n+1$ and the last basic unit is not a special unit can be obtained from the partitions with length $3n+3$ and the last basic unit being not a special unit by omitting the parts λ_{3n+3} and λ_{3n+2}. Therefore, the generating function, for each $0 \leq d \leq n-2$, for partitions of length $3n+1$ and the last basic unit being not a special unit can be obtained from the generating function for partitions with length $3n+3$ and the last basic unit being not a special unit by omitting the factor

$$\frac{q^{3n+3}}{1-q^{3n+3}}.$$

These correspond to the terms (15) to (18) in Lemma 2.4.

We complete the proof of Lemma 2.4.

We next consider the *Case B*.

Lemma 2.5 The generating function for the number of partitions with the given length and alternating sum types $(\Sigma, 2)$ and belonging to *Case B* is the sum of the following four terms.

$$\frac{z^{n+1}q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}}\frac{n}{z}\,(n\geq 1),\ \text{for length } 3n+3, \tag{29}$$

$$\frac{z^{n+1}q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}}\frac{q^{3n+3}}{z}\,(n\geq 0),\ \text{for length } 3n+3, \tag{30}$$

$$\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_{n}}\frac{n}{z}\,(n\geq 1),\ \text{for length } 3n+1, \tag{31}$$

$$\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_{n}}\frac{q^{3n+3}}{z}\,(n\geq 0),\ \text{for length } 3n+2. \tag{32}$$

Proof of Lemma 2.5. In *Case B*, each partition has only one special unit. We still consider three partitions classes by their length $3n+3$, $3n+2$ and $3n+1$, where $n \geq 0$. In each class, the special unit can be the last basic unit or not.

Length $3n+3$. λ has the form:

$$\lambda_1 > \lambda_2 = \lambda_3 > \cdots = \lambda_{3k-3} > \lambda_{3k-2} \geq \lambda_{3k-1} \overset{2}{>} \lambda_{3k} \geq \lambda_{3k+1} > \cdots = \lambda_{3n+3}, (1 \leq k \leq n),$$

or the form:

$$\lambda_1 > \lambda_2 = \lambda_3 > \cdots = \lambda_{3n} > \lambda_{3n+1} \geq \lambda_{3n+2} \overset{2}{>} \lambda_{3n+3}.$$

The generating function for the former is

$$\begin{aligned}
&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3k-3}}{1-q^{3k-3}}\frac{1}{1-zq^{3k-2}}q^{6k-2}\frac{1}{1-q^{3k}}\cdot\frac{zq^{3k+1}}{1-zq^{3k+1}}\cdots\frac{zq^{3n+1}}{1-zq^{3n+1}}\frac{q^{3n+3}}{1-q^{3n+3}}\\
=&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{zq^{3n+1}}{1-zq^{3n+1}}\frac{q^{3n+3}}{1-q^{3n+3}}\frac{1}{z}\\
=&\frac{z^{n+1}q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}}\frac{1}{z},\quad (1\leq k\leq n).
\end{aligned}$$

The special unit can be any k^{th} basic unit, $1 \leq k \leq n$, so we get the term (29) in Lemma 2.5 by adding them. The generating function for the later is

$$\begin{aligned}&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n}}{1-q^{3n}}\frac{zq^{3n+1}}{1-zq^{3n+1}}\frac{1}{zq^{3n+1}}q^{6n+4}\frac{q^{3n+3}}{1-q^{3n+3}}\\=&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{zq^{3n+1}}{1-zq^{3n+1}}\frac{q^{3n+3}}{1-q^{3n+3}}\frac{q^{3n+3}}{z}\\=&\frac{z^{n+1}q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}}\frac{q^{3n+3}}{z},\quad(n\ge 0).\end{aligned}$$

We get the term (30) in lemma 2.5.

Length $3n+2$. Special unit in λ must be the last basic unit. We note that such partitions can be obtained from partitions belonging to the *Case B* with length $3n+3$ and the special unit being the last basic unit by omitting the part λ_{3n+3} and Letting $\lambda_{3n+2}=2$. Hence the generating function for partitions belong to the *Case B* with length $3n+2$ and the special unit being the last basic unit can be obtained from the generating function for partitions belong to *Case B* with length $3n+3$ and the special is the last basic unit by omitting the factor

$$\frac{q^{3n+3}}{1-q^{3n+3}}.$$

So we get the term (32) in Lemma 2.5.

Length $3n+1$. In this case, the special unit can not be the last basic unit. As the analysis above, we find that such partitions can be obtained from partitions belonging to the *Case B* with length $3n+3$ and the special unit being not the last basic unit by omitting the part λ_{3n+3} and the part λ_{3n+2}. Hence the generating function for partitions belong to the *Case B* with length $3n+1$ and the special unit being not the last basic unit can be obtained from the generating function for partitions belong to *Case B* with length $3n+3$ and the special being not the last basic unit by omitting the factor

$$\frac{q^{3n+3}}{1-q^{3n+3}}.$$

It corresponds the term (31) in Lemma 2.5.

We complete the proof of Lemma 2.5.

Now the generating function for $a(\Sigma,2;n)$ will be the sums of all terms in Lemma 2.2, Lemma 2.3, Lemma 2.4 and Lemma 2.5 involving $3n+3$, $3n+2$, and $3n+1$, and adds them together where n runs over the ranges indicating in these lemmas. We firstly compute the sum of terms only involving $\frac{1}{z}$. It is

$$\begin{aligned}&\sum_{n\ge1}\frac{z^{n+1}q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}}\frac{n}{z}+\sum_{n\ge1}\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_{n}}\frac{n}{z}+\sum_{n\ge0}\frac{z^{n+1}q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}}\frac{q^{3n+3}}{z}\\&+\sum_{n\ge0}\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_{n}}\frac{q^{3n+3}}{z}+\sum_{n\ge2}\frac{z^{n+1}q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}}\sum_{k=1}^{n-1}\frac{1-q^{3k}}{zq^{3k}}+\sum_{n\ge1}\frac{z^{n+1}q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}}\\&\cdot\frac{q^3(1-q^{3n})}{z}+\sum_{n\ge1}\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_{n}}\frac{q^3(1-q^{3n})}{z}+\sum_{n\ge2}\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_{n}}\sum_{k=1}^{n-1}\frac{1-q^{3k}}{zq^{3k}}\\=&\sum\text{terms}(n\ge2)+\sum\text{ terms }(n=1)+\sum\text{terms }(n=0).\end{aligned}$$

After using the identity

$$\sum_{n\ge2}\frac{z^{n+1}q^{3n^2+7n+4}}{(zq;q^3)_{n+1}(q^3;q^3)_{n+1}}A+\sum_{n\ge2}\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_{n}}A=\sum_{n\ge2}\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_{n}}\frac{1}{1-q^{3n+3}}A,$$

we have

$$\begin{aligned}\sum\text{terms}(n\ge2)=&\sum_{n\ge2}\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_{n}}\frac{1}{1-q^{3n+3}}\frac{n}{z}+\sum_{n\ge2}\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_{n}}\cdot\frac{1}{1-q^{3n+3}}\frac{q^{3n+3}}{z}\\&+\sum_{n\ge2}\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_{n}}\frac{1}{1-q^{3n+3}}\frac{q^3(1-q^{3n})}{z}\\&+\sum_{n\ge2}\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_{n}}\frac{1}{1-q^{3n+3}}\sum_{k=1}^{n-1}\frac{1-q^{3k}}{zq^{3k}}\end{aligned}$$

$$
\begin{aligned}
&=\sum_{n\geq 2}\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_n}\frac{1}{1-q^{3n+3}}\cdot\left(\frac{n}{z}+\frac{q^3(1-q^{3n})}{z}+\frac{q^{3n+3}}{z}+\sum_{k=1}^{n-1}\frac{1-q^{3k}}{zq^{3k}}\right)\\
&=\sum_{n\geq 2}\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{q^{3n}}{1-q^{3n}}\frac{1}{1-zq^{3n+1}}\left(\frac{nq^{3n+1}}{1-q^{3n+3}}+\frac{q^{3n+4}(1-q^{3n})}{1-q^{3n+3}}\right.\\
&\quad\left.+\frac{q^{6n+4}}{1-q^{3n+3}}+\frac{q^4(1-q^{3n-3})}{(1-q^{3n+3})(1-q^3)}+\frac{(1-n)q^{3n+1}}{1-q^{3n+3}}\right)\\
&=\left(\sum_{n\geq 2}\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{q^{3n}}{1-q^{3n}}\frac{1}{1-zq^{3n+1}}\right)\frac{q^4}{1-q^3}.
\end{aligned}
$$

And

$$
\begin{aligned}
&\sum\text{terms}(n=0)+\sum\text{terms}(n=1)\\
=&\frac{1}{1-zq}\frac{q^4}{1-q^3}+\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{1}{1-zq^4}\left(\frac{q^{10}}{1-q^6}+\frac{q^4}{1-q^6}\right)+\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\frac{1}{1-q^6}\frac{q^3(1-q^3)}{z}\\
=&\frac{1}{1-zq}\frac{q^4}{1-q^3}+\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{1}{1-zq^4}\frac{q^4}{1-q^3}.
\end{aligned}
$$

Combine it with the last equality above, we get the sum only involving $\frac{1}{z}$ is

$$
\left(\sum_{n\geq 0}\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{q^{3n}}{1-q^{3n}}\frac{1}{1-zq^{3n+1}}\right)\frac{q^4}{1-q^3}. \tag{33}
$$

Now we compute the sum involving $\frac{1}{z^2}$ in Lemma 2.2 lemma 2.3 and Lemma 2.4. After using the identity

$$
\left(\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_n}\frac{q^{3n+3}}{1-q^{3n+3}}\right)A+\left(\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_n}\right)A=\left(\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_n}\frac{1}{1-q^{3n+3}}\right)A.
$$

We find there are four sums involving $\frac{1}{z^2}$, three sums of them are

$$
\sum_{n\geq 1}\left(\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_n}\frac{1}{1-q^{3n+3}}\right)\frac{q^2}{z^2}, \tag{34}
$$

$$
\sum_{n\geq 2}\left(\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_n}\frac{1}{1-q^{3n+3}}\right)\sum_{k=1}^{n-1}\frac{1}{z^2q^{3k+1}}, \tag{35}
$$

$$
\sum_{n\geq 2}\left(\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_n}\frac{1}{1-q^{3n+3}}\right)\frac{1}{z^2}\left(\frac{1}{q}+\frac{1}{q^4}+\cdots+\frac{1}{q^{3n-5}}\right), \tag{36}
$$

and the fourth is the sum of the following $n-2$ terms:

$$
\sum_{n\geq 3}\left(\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_n}\frac{1}{1-q^{3n+3}}\right)\sum_{k=1}^{n-2}\frac{1}{z^2q^{6k+4}} \tag{37}
$$

$$
+\sum_{n\geq 4}\left(\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_n}\frac{1}{1-q^{3n+3}}\right)\sum_{k=1}^{n-3}\frac{1}{z^2q^{6k+7}} \tag{38}
$$

$$+\sum_{n\geq 5}\left(\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_n}\frac{1}{1-q^{3n+3}}\right)\sum_{k=1}^{n-4}\frac{1}{z^2q^{6k+10}} \tag{39}$$
$$+\dots$$
$$+\sum_{n\geq 3}\left(\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_n}\frac{1}{1-q^{3n+3}}\right)\sum_{k=1}^{n-(n-1)}\frac{1}{z^2q^{6k+3n-5}}. \tag{40}$$

Where the last term (40) corresponds to the case that the distance between two special units is $n-2$ and the last unit is not a special unit, there is only one case, corresponding to $k=1$, hence $n>2$. So the range for sum is $n\geq 3$ in (40). Note that the case of the distance zero already appears in (37) (the term of $k=n-2$). Now

$$\begin{aligned}
\text{The first sum } &= \sum_{n\geq 1}\left(\frac{z^{n+1}q^{3n^2+4n+1}}{(zq;q^3)_{n+1}(q^3;q^3)_n}\frac{1}{1-q^{3n+3}}\right)\frac{q^2}{z^2}\\
&=\sum_{n\geq 1}\left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{zq^{3n-2}}{1-zq^{3n-2}}\frac{q^{3n}}{1-q^{3n}}\frac{zq^{3n+1}}{1-zq^{3n+1}}\frac{1}{1-q^{3n+3}}\right)\frac{q^2}{z^2}\\
&=\sum_{n\geq 1}\left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n-3}}{1-q^{3n-3}}\frac{1}{1-zq^{3n-2}}\right)\cdot\frac{q^{3n-2}}{1}\cdot\frac{q^{3n}}{1-q^{3n}}\cdot\frac{1}{1-zq^{3n+1}}\cdot\frac{q^{3n+1}}{1-q^{3n+3}}\cdot\frac{q^2}{1}\\
&=\sum_{n\geq 1}\left(\frac{zq}{1-zq}\cdots\frac{q^{3n-3}}{1-q^{3n-3}}\frac{1}{1-zq^{3n-2}}\right)\cdot\frac{q^{3n-2}}{1}\cdot\frac{q^{3n}}{1-q^{3n}}\cdot(1+zq^{3n+1}+\dots)\cdot\frac{q^{3n+1}}{1-q^{3n+3}}\cdot\frac{q^2}{1}\\
&=\sum_{n\geq 1}\left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n-3}}{1-q^{3n-3}}\frac{1}{1-zq^{3n-2}}\right)\frac{q^{3n-2}}{1}\cdot\frac{q^{3n}}{1-q^{3n}}\cdot 1\cdot\frac{q^{3n+1}}{1-q^{3n+3}}\cdot\frac{q^2}{1}+\\
&\quad\sum_{n\geq 1}\left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n-3}}{1-q^{3n-3}}\frac{1}{1-zq^{3n-2}}\right)\cdot\frac{q^{3n-2}}{1}\cdot\frac{q^{3n}}{1-q^{3n}}\cdot\frac{zq^{3n+1}}{1-zq^{3n+1}}\cdot\frac{q^{3n+1}}{1-q^{3n+3}}\cdot\frac{q^2}{1}\\
&=\sum_{n\geq 1}\left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n-3}}{1-q^{3n-3}}\frac{1}{1-zq^{3n-2}}\right)\frac{q^{3n-2}}{1}\cdot\frac{q^{3n}}{1-q^{3n}}\cdot 1\cdot\frac{q^{3n+1}}{1-q^{3n+3}}\cdot\frac{q^2}{1}\\
&\quad+\sum_{n\geq 1}\left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n}}{1-q^{3n}}\frac{1}{1-zq^{3n+1}}\right)\frac{q^{3n+1}}{1}\cdot\frac{q^{3n+1}}{1-q^{3n+3}}\cdot\frac{q^2}{1}\\
&=\sum_{n\geq 1}\left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n-3}}{1-q^{3n-3}}\frac{1}{1-zq^{3n-2}}\right)\frac{q^{9n+1}}{(1-q^{3n})(1-q^{3n+3})}\\
&\quad+\sum_{n\geq 1}\left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n}}{1-q^{3n}}\frac{1}{1-zq^{3n+1}}\right)\frac{q^{6n+4}}{1-q^{3n+3}}.
\end{aligned}$$

Let

$$A_1=\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n-3}}{1-q^{3n-3}}\frac{1}{1-zq^{3n-2}},$$

$$A_2=\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n}}{1-q^{3n}}\frac{1}{1-zq^{3n+1}}.$$

$$\begin{aligned}
\text{The second sum } &= \sum_{n\geq 2}\left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{zq^{3n+1}}{1-zq^{3n+1}}\frac{1}{1-q^{3n+3}}\right)\sum_{k=1}^{n-1}\frac{1}{z^2q^{3k+1}}\\
&=\sum_{n\geq 2}\left(\frac{zq}{1-zq}\cdots\frac{q^{3n-3}}{1-q^{3n-3}}\frac{1}{1-zq^{3n-2}}\right)\frac{q^{3n-2}}{1}\cdot\frac{q^{3n}}{1-q^{3n}}\cdot\frac{1}{1-zq^{3n+1}}\cdot\frac{q^{3n+1}}{1-q^{3n+3}}\cdot\sum_{k=1}^{n-1}\frac{1}{q^{3k+1}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n\geq 2} A_1 \frac{q^{3n-2}}{1} \cdot \frac{q^{3n}}{1-q^{3n}} \cdot (1+zq^{3n+1}+\dots) \cdot \frac{q^{3n+1}}{1-q^{3n+3}} \cdot \sum_{k=1}^{n-1} \frac{1}{q^{3k+1}} \\
&= \sum_{n\geq 2} A_1 \frac{q^{3n-2}}{1} \cdot \frac{q^{3n}}{1-q^{3n}} \cdot 1 \cdot \frac{q^{3n+1}}{1-q^{3n+3}} \cdot \left(\frac{1}{q^4}+\frac{1}{q^7}+\dots+\frac{1}{q^{3n-2}}\right) \\
&\quad + \sum_{n\geq 2} A_1 \frac{q^{3n-2}}{1} \cdot \frac{q^{3n}}{1-q^{3n}} \cdot \frac{zq^{3n+1}}{1-zq^{3n+1}} \cdot \frac{q^{3n+1}}{1-q^{3n+3}} \cdot \left(\frac{1}{q^4}+\frac{1}{q^7}+\dots+\frac{1}{q^{3n-2}}\right) \\
&= \sum_{n\geq 2} A_1 \frac{1}{(1-q^{3n})(1-q^{3n+3})} \cdot \left(q^{6n+1}+q^{6n+4}+\dots+q^{9n-5}\right) \\
&\quad + \sum_{n\geq 2} A_2 \frac{1}{1-q^{3n+3}} \cdot \left(q^{3n+4}+q^{3n+7}+\dots+q^{6n-2}\right) \\
&= \sum_{n\geq 2} \left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n-3}}{1-q^{3n-3}}\frac{1}{1-zq^{3n-2}}\right) \frac{q^{6n+1}(1-q^{3n-3})}{(1-q^{3n})(1-q^{3n+3})(1-q^3)} \\
&\quad + \sum_{n\geq 2} \left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n}}{1-q^{3n}}\frac{1}{1-zq^{3n+1}}\right) \frac{q^{3n+4}(1-q^{3n-3})}{(1-q^{3n+3})(1-q^3)}.
\end{aligned}
$$

The third sum
$$
\begin{aligned}
&= \sum_{n\geq 2} \left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{zq^{3n+1}}{1-zq^{3n+1}}\frac{1}{1-q^{3n+3}}\right) \cdot \frac{1}{z^2}\left(\frac{1}{q}+\frac{1}{q^4}+\dots+\frac{1}{q^{3n-5}}\right) \\
&= \sum_{n\geq 2} \left(\frac{zq}{1-zq}\cdots\frac{1}{1-zq^{3n-2}}\right) \frac{q^{3n-2}}{1} \cdot \frac{q^{3n}}{1-q^{3n}} \cdot \frac{1}{1-zq^{3n+1}} \cdot \frac{q^{3n+1}}{1-q^{3n+3}} \cdot \left(\frac{1}{q}+\frac{1}{q^4}+\dots+\frac{1}{q^{3n-5}}\right) \\
&= \sum_{n\geq 2} A_1 \frac{q^{3n-2}}{1} \cdot \frac{q^{3n}}{1-q^{3n}} \cdot (1+zq^{3n+1}+\dots) \cdot \frac{q^{3n+1}}{1-q^{3n+3}} \cdot \left(\frac{1}{q}+\frac{1}{q^4}+\dots+\frac{1}{q^{3n-5}}\right) \\
&= \sum_{n\geq 2} A_1 \frac{q^{3n-2}}{1} \cdot \frac{q^{3n}}{1-q^{3n}} \cdot 1 \cdot \frac{q^{3n+1}}{1-q^{3n+3}} \cdot \left(\frac{1}{q}+\frac{1}{q^4}+\dots+\frac{1}{q^{3n-5}}\right) + \\
&\quad \sum_{n\geq 2} A_1 \frac{q^{3n-2}}{1} \cdot \frac{q^{3n}}{1-q^{3n}} \cdot \frac{zq^{3n+1}}{1-zq^{3n+1}} \cdot \frac{q^{3n+1}}{1-q^{3n+3}} \cdot \left(\frac{1}{q}+\frac{1}{q^4}+\dots+\frac{1}{q^{3n-5}}\right) \\
&= \sum_{n\geq 2} A_1 \frac{1}{(1-q^{3n})(1-q^{3n+3})} \cdot \left(q^{6n+4}+q^{6n+4}+\dots+q^{9n-2}\right) \\
&\quad + \sum_{n\geq 2} A_2 \frac{1}{1-q^{3n+3}} \cdot \left(q^{3n+7}+q^{3n+7}+\dots+q^{6n+1}\right) \\
&= \sum_{n\geq 2} \left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n-3}}{1-q^{3n-3}}\frac{1}{1-zq^{3n-2}}\right) \frac{q^{6n+4}(1-q^{3n-3})}{(1-q^{3n})(1-q^{3n+3})(1-q^3)} \\
&\quad + \sum_{n\geq 2} \left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n}}{1-q^{3n}}\frac{1}{1-zq^{3n+1}}\right) \frac{q^{3n+7}(1-q^{3n-3})}{(1-q^{3n+3})(1-q^3)}.
\end{aligned}
$$

Before computing the fourth sum, we first compute the d^{th} term in the fourth sum:

The d^{th} term
$$
\begin{aligned}
&= \sum_{n\geq d+2} \left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{zq^{3n+1}}{1-zq^{3n+1}}\frac{1}{1-q^{3n+3}}\right) \sum_{k=1}^{n-d-1} \frac{1}{z^2}\frac{1}{q^{6k+3d+1}} \\
&= \sum_{n\geq d+2} \left(\frac{zq}{1-zq}\cdots\frac{1}{1-zq^{3n-2}}\right) \frac{q^{3n-2}}{1} \cdot \frac{q^{3n}}{1-q^{3n}} \cdot \frac{1}{1-zq^{3n+1}} \cdot \frac{q^{3n+1}}{1-q^{3n+3}} \cdot \sum_{k=1}^{n-d-1} \frac{1}{q^{6k+3d+1}}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n \geq d+2} A_1 \frac{q^{3n-2}}{1} \cdot \frac{q^{3n}}{1-q^{3n}} \cdot (1 + zq^{3n+1} + \dots) \cdot \frac{q^{3n+1}}{1-q^{3n+3}} \cdot \sum_{k=1}^{n-d-1} \frac{1}{q^{6k+3d+1}} \\
&= \sum_{n \geq d+2} A_1 \frac{q^{3n-2}}{1} \cdot \frac{q^{3n}}{1-q^{3n}} \cdot 1 \cdot \frac{q^{3n+1}}{1-q^{3n+3}} \cdot \sum_{k=1}^{n-d-1} \frac{1}{q^{6k+3d+1}} \\
&\quad + \sum_{n \geq d+2} A_1 \frac{q^{3n-2}}{1} \cdot \frac{q^{3n}}{1-q^{3n}} \cdot \frac{zq^{3n+1}}{1-zq^{3n+1}} \cdot \frac{q^{3n+1}}{1-q^{3n+3}} \cdot \sum_{k=1}^{n-d-1} \frac{1}{q^{6k+3d+1}} \\
&= \sum_{n \geq d+2} A_1 \frac{1}{(1-q^{3n})(1-q^{3n+3})} \cdot \left(q^{3n+3d+4} + q^{3n+3d+10} + \dots + q^{9n-3d-8}\right) \\
&\quad + \sum_{n \geq d+2} A_2 \frac{1}{1-q^{3n+3}} \cdot \left(q^{3d+7} + q^{3d+13} + \dots + q^{6n-3d-5}\right) \\
&= \sum_{n \geq d+2} \left(\frac{zq}{1-zq} \frac{q^3}{1-q^3} \cdots \frac{q^{3n-3}}{1-q^{3n-3}} \frac{1}{1-zq^{3n-2}} \right) \cdot \frac{q^{3n+3d+4}(1-q^{6n-6d-6})}{(1-q^{3n})(1-q^{3n+3})(1-q^6)} \\
&\quad + \sum_{n \geq d+2} \left(\frac{zq}{1-zq} \frac{q^3}{1-q^3} \cdots \frac{q^{3n}}{1-q^{3n}} \frac{1}{1-zq^{3n+1}} \right) \frac{q^{3d+7}(1-q^{6n-6d-6})}{(1-q^{3n+3})(1-q^6)}.
\end{aligned}
$$

Therefore, the fourth sum

$$
\begin{aligned}
&= \sum_{n \geq 3} \left(\frac{zq}{1-zq} \frac{q^3}{1-q^3} \cdots \frac{q^{3n-3}}{1-q^{3n-3}} \frac{1}{1-zq^{3n-2}} \right) \cdot \sum_{d=1}^{n-2} \frac{q^{3n+3d+4}(1-q^{6n-6d-6})}{(1-q^{3n})(1-q^{3n+3})(1-q^6)} \\
&\quad + \sum_{n \geq 3} \left(\frac{zq}{1-zq} \frac{q^3}{1-q^3} \cdots \frac{q^{3n}}{1-q^{3n}} \frac{1}{1-zq^{3n+1}} \right) \sum_{d=1}^{n-2} \frac{q^{3d+7}(1-q^{6n-6d-6})}{(1-q^{3n+3})(1-q^6)} \\
&= \sum_{n \geq 3} \left(\frac{zq}{1-zq} \frac{q^3}{1-q^3} \cdots \frac{q^{3n-3}}{1-q^{3n-3}} \frac{1}{1-zq^{3n-2}} \right) \cdot \frac{q^{3n+7}(1-q^{3n-3})(1-q^{3n-6})}{(1-q^{3n})(1-q^{3n+3})(1-q^6)(1-q^3)} \\
&\quad + \sum_{n \geq 3} \left(\frac{zq}{1-zq} \frac{q^3}{1-q^3} \cdots \frac{q^{3n}}{1-q^{3n}} \frac{1}{1-zq^{3n+1}} \right) \frac{q^{10}(1-q^{3n-3})(1-q^{3n-6})}{(1-q^{3n+3})(1-q^6)(1-q^3)}.
\end{aligned}
$$

Now the sum of four sums is the first sum$+$the second sum$+$ the third sum$+$the fourth sum indicated above. Since in the four sums, the common range of n is $n \geq 3$, so we first consider the four sums over all $n \geq 3$, and then consider terms corresponding to $n = 1$ and $n = 2$.

The first sum restrict to $n \geq 3$ + the second sum restrict to $n \geq 3$
+ the third sum restrict to $n \geq 3$ + the fourth sum restrict to $n \geq 3$

$$
\begin{aligned}
&= \sum_{n \geq 3} \left(\frac{zq}{1-zq} \frac{q^3}{1-q^3} \cdots \frac{q^{3n-3}}{1-q^{3n-3}} \frac{1}{1-zq^{3n-2}} \right) \frac{q^{9n+1}}{(1-q^{3n})(1-q^{3n+3})} \\
&\quad + \sum_{n \geq 3} \left(\frac{zq}{1-zq} \frac{q^3}{1-q^3} \cdots \frac{q^{3n}}{1-q^{3n}} \frac{1}{1-zq^{3n+1}} \right) \frac{q^{6n+4}}{1-q^{3n+3}} \\
&\quad + \sum_{n \geq 3} \left(\frac{zq}{1-zq} \frac{q^3}{1-q^3} \cdots \frac{q^{3n-3}}{1-q^{3n-3}} \frac{1}{1-zq^{3n-2}} \right) \frac{q^{6n+1}(1-q^{3n-3})}{(1-q^{3n})(1-q^{3n+3})(1-q^3)} \\
&\quad + \sum_{n \geq 3} \left(\frac{zq}{1-zq} \frac{q^3}{1-q^3} \cdots \frac{q^{3n}}{1-q^{3n}} \frac{1}{1-zq^{3n+1}} \right) \frac{q^{3n+4}(1-q^{3n-3})}{(1-q^{3n+3})(1-q^3)} \\
&\quad + \sum_{n \geq 3} \left(\frac{zq}{1-zq} \frac{q^3}{1-q^3} \cdots \frac{q^{3n-3}}{1-q^{3n-3}} \frac{1}{1-zq^{3n-2}} \right) \frac{q^{6n+4}(1-q^{3n-3})}{(1-q^{3n})(1-q^{3n+3})(1-q^3)}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n\geq 3}\left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n}}{1-q^{3n}}\frac{1}{1-zq^{3n+1}}\right)\frac{q^{3n+7}(1-q^{3n-3})}{(1-q^{3n+3})(1-q^3)}\\
&+\sum_{n\geq 3}\left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n-3}}{1-q^{3n-3}}\frac{1}{1-zq^{3n-2}}\right)\cdot\frac{q^{3n+7}(1-q^{3n-3})(1-q^{3n-6})}{(1-q^{3n})(1-q^{3n+3})(1-q^6)(1-q^3)}\\
&+\sum_{n\geq 3}\left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n}}{1-q^{3n}}\frac{1}{1-zq^{3n+1}}\right)\frac{q^{10}(1-q^{3n-3})(1-q^{3n-6})}{(1-q^{3n+3})(1-q^6)(1-q^3)}\\
=&\sum_{n\geq 3}A_1\left(\frac{q^{9n+1}}{(1-q^{3n})(1-q^{3n+3})}+\frac{q^{6n+1}(1-q^{3n-3})}{(1-q^{3n})(1-q^{3n+3})(1-q^3)}\right.\\
&\left.+\frac{q^{6n+4}(1-q^{3n-3})}{(1-q^{3n})(1-q^{3n+3})(1-q^3)}+\frac{q^{3n+7}(1-q^{3n-3})(1-q^{3n-6})}{(1-q^3)(1-q^6)(1-q^{3n})(1-q^{3n+3})}\right)\\
&+\sum_{n\geq 3}A_2\left(\frac{q^{6n+4}}{1-q^{3n+3}}+\frac{q^{3n+4}(1-q^{3n-3})}{(1-q^{3n+3})(1-q^3)}+\frac{q^{3n+7}(1-q^{3n-3})}{(1-q^{3n+3})(1-q^3)}+\frac{q^{10}(1-q^{3n-3})(1-q^{3n-6})}{(1-q^3)(1-q^6)(1-q^{3n+3})}\right)\\
=&\sum_{n\geq 3}A_1\frac{q^{3n+7}}{(1-q^3)(1-q^6)}+\sum_{n\geq 3}A_2\frac{q^{10}(1-q^{3n})}{(1-q^3)(1-q^6)}.
\end{aligned}
$$

But

$$
\sum_{n\geq 3}A_1\frac{q^{3n+7}}{(1-q^3)(1-q^6)}=\sum_{n\geq 2}A_2\frac{q^{3n+10}}{(1-q^3)(1-q^6)},
$$

Hence

$$
\begin{aligned}
&\text{The first sum restrict to } n\geq 3+\text{the second sum restrict to } n\geq 3\\
&+\text{ the third sum restrict to } n\geq 3+\text{the fourth sum restrict to } n\geq 3\\
=&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\frac{q^6}{1-q^6}\frac{1}{1-zq^7}\frac{q^{16}}{(1-q^3)(1-q^6)}+\sum_{n\geq 3}A_2\frac{q^{3n+10}}{(1-q^3)(1-q^6)}+\sum_{n\geq 3}A_2\frac{q^{10}(1-q^{3n})}{(1-q^3)(1-q^6)}\\
=&\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\frac{q^6}{1-q^6}\frac{1}{1-zq^7}\frac{q^{16}}{(1-q^3)(1-q^6)}+\sum_{n\geq 3}A_2\frac{q^{10}}{(1-q^3)(1-q^6)}.
\end{aligned}
\tag{41}
$$

Now we consider the sum of terms $n\leq 2$, their sum is

$$
\begin{aligned}
&\text{the first sum restrict to } n<3+\text{the second sum restrict to } n<3\\
&+\text{ the third sum restrict to } n<3+\text{the fourth sum restrict to } n<3\\
=&\sum_{n=1}^{2}\left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n-3}}{1-q^{3n-3}}\frac{1}{1-zq^{3n-2}}\right)\frac{q^{9n+1}}{(1-q^{3n})(1-q^{3n+3})}\\
&+\sum_{n=1}^{2}\left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n}}{1-q^{3n}}\frac{1}{1-zq^{3n+1}}\right)\frac{q^{6n+4}}{1-q^{3n+3}}\\
&+\sum_{n=2}^{2}\left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n-3}}{1-q^{3n-3}}\frac{1}{1-zq^{3n-2}}\right)\frac{q^{6n+1}(1-q^{3n-3})}{(1-q^{3n})(1-q^{3n+3})(1-q^3)}\\
&+\sum_{n=2}^{2}\left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n}}{1-q^{3n}}\frac{1}{1-zq^{3n+1}}\right)\frac{q^{3n+4}(1-q^{3n-3})}{(1-q^{3n+3})(1-q^3)}\\
&+\sum_{n=2}^{2}\left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n-3}}{1-q^{3n-3}}\frac{1}{1-zq^{3n-2}}\right)\frac{q^{6n+4}(1-q^{3n-3})}{(1-q^{3n})(1-q^{3n+3})(1-q^3)}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n=2}^{2}\left(\frac{zq}{1-zq}\frac{q^3}{1-q^3}\cdots\frac{q^{3n}}{1-q^{3n}}\frac{1}{1-zq^{3n+1}}\right)\frac{q^{3n+7}(1-q^{3n-3})}{(1-q^{3n+3})(1-q^3)}\\
=&\frac{1}{1-zq}\frac{q^{10}}{(1-q^3)(1-q^6)}+\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{1}{1-zq^4}\frac{q^{19}}{(1-q^6)(1-q^9)}+\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{1}{1-zq^4}\frac{q^{10}}{1-q^6}\\
&+\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\frac{q^6}{1-q^6}\frac{1}{1-zq^7}\frac{q^{16}}{1-q^9}+\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{1}{1-zq^4}\frac{q^{13}}{(1-q^6)(1-q^9)}\\
&+\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\frac{q^6}{1-q^6}\frac{1}{1-zq^7}\frac{q^{10}}{1-q^9}+\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{1}{1-zq^4}\frac{q^{16}}{(1-q^6)(1-q^9)}\\
&+\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\frac{q^6}{1-q^6}\frac{1}{1-zq^7}\frac{q^{13}}{1-q^9}\\
=&\left(\frac{1}{1-zq}+\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{1}{1-zq^4}\right)\frac{q^{10}}{(1-q^3)(1-q^6)}+\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\frac{q^6}{1-q^6}\frac{1}{1-zq^7}\frac{q^{10}}{1-q^3}. \qquad (42)
\end{aligned}
$$

Combine (41) with (42), we get the sum only involving $\frac{1}{z^2}$ is

$$
\left(\sum_{n\geq 0}\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{q^{3n}}{1-q^{3n}}\frac{1}{1-zq^{3n+1}}\right)\frac{q^{10}}{(1-q^3)(1-q^6)}. \qquad (43)
$$

Therefore, the generating function for the partitions with repeated times ≤ 2 and alternating sum types $(\Sigma, 2)$ is

$$
\begin{aligned}
&\sum_{\Sigma,n\geq 0}a(\Sigma,2;n)z^{\Sigma}q^n=\text{the sums involving}\frac{1}{z}+\text{ the sums involving}\frac{1}{z^2}\\
=&(33)+(43)\\
=&\left(\sum_{n\geq 0}\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{q^{3n}}{1-q^{3n}}\frac{1}{1-zq^{3n+1}}\right)\frac{q^4}{(1-q^3)(1-q^6)}.
\end{aligned}
$$

But the function

$$
\sum_{n\geq 0}\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{q^{3n}}{1-q^{3n}}\frac{1}{1-zq^{3n+1}}
$$

is exactly the generating function of $a(\Sigma, 0; n)$! Since the general term

$$
\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{q^{3n}}{1-q^{3n}}\frac{1}{1-zq^{3n+1}}
$$

generates all partitions with pure type $(\Sigma, 0)$ with length $3n$ or $3n+1$: the generating function for partitions with pure type $(\Sigma, 0)$ and length $3n$ (by considering their conjugates) corresponds to

$$
\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{q^{3n}}{1-q^{3n}}
$$

and the generating function for partitions with pure type $(\Sigma, 0)$ and length $3n+1$ corresponds to

$$
\frac{zq}{1-zq}\frac{q^3}{1-q^3}\frac{zq^4}{1-zq^4}\cdots\frac{q^{3n}}{1-q^{3n}}(zq^{3n+1}+z^2q^{6n+2}+\dots).
$$

Hence we proved $a(\Sigma, 2; n)$ satisfies the same relation as $b(\Sigma, 2; n)$.

The proof is completed.

3. Conclusion

We use q-series and generating function theories to provide an analytic style proof for $m = 3$ and alternative sum type $(\Sigma, 2)$. Our method is verifying the generating functions of both sides satisfying the same recurrences. According to this method, we can also get more general results, but because the counting process is more complicated, we will not give the proof process here, and those who are interested can try to count.

References

[1] H. L. Alder, *Partition identities - from Euler to the present*, Am. Math. Monthly 76 (1969), 733-746.

[2] G. E. Andrews, *The theory of partitions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998.

[3] G. E. Andrews, *On generalizations of Euler's partition theorem*, Michigan Math. J. 1 (1966), 491–498.

[4] W. Y. C. Chen, H. Y. Gao, K. Q. Ji, M. Y. X. Li, *A unification of two refinements of Euler's partition theorem*, Ramanujan J. 23 (2010), 137-149.

[5] S. Fu, D. Tang, A. J. Yee *A Lecture Hall Theorem for m-Falling Partitions*, Annals of Combinatorics, 23 (2019), Number 3-4, 749–764.

[6] D. Kim and A. J. Yee, *A note on partitions into distinct parts and odd parts*, Ramanujan J. 3 (1999) 227–231.

[7] I. Pak and A. Postnikov, *A generalization of Sylvester's Identity*, Discrete Math. 178 (1998), 277-281.

[8] C. D. Savage, A. J. Yee, *Euler's partition theorem and the combinatorics of l-sequences*. J. Combin. Theory Ser. A 115 (2008), 967-996.

[9] X. Xiong and W. J. Keith, *Eulers partition theorem for all moduli and new companions to Rogers Ramanujan Andrews-Gordon identities*. Ramanujan J. 49 (2019), no. 3 , 555-565.

[10] Jiang Zeng, *The q-variations of Sylvester's bijection between odd and strict partitions*, Ramanujan J. 9 (2005), 289-303.

[11] K. Alladi, *Partitions with non-repeating odd parts and combinatorial identities*, Ann. Comb. 20, 1-20 (2016)

[12] I. Pak, *Partition bijections*, a survey, Ramanujan J. 12, 5-75 (2006).

[13] K. Alladi, and A. Berkovich, *New weighted Rogers-Ramanujan partition theorems and their implications*, Trans. Amer. Math. Soc. 354 (7), 2557-2577 (2002).

[14] A. J. Yee, *On the combinatorics of lecture hall partitions*, Ramanujan J. 5, 247-262 (2001).

[15] A. J. Yee, *On the refined lecture hall theorem*, Discrete Math. 248, 293-298 (2002).