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Noetherity of a Dirac Delta-Extension for a Noether Operator

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Abstract: The main goal of this work is to establish the extension of a noether operator defined by a third kind singular integral equation in a special class of generalized functions and to investigate the noetherity of the extended operator. The initial considered operator has been investigated for its noetherity nature in our previous works. Special approach has been developed and applied when constructing noether theory for the mentioned operator by using the concept of Taylor derivatives for continuous functions to achieve noetherity. We realize the extension of the initial noether operator defined onto the class of coninuous functions, having first continous derivative and taking the value zero at the left boundary point of the closed interval by adding a finite number of Dirac delta functions and it Taylor derivatives of some order. Then, we investigate the noetherity of the extended initial noether operator when we realize the finite dimensional extension, taking the unknown function from a special space of generalized functions. For this aim, we will use the direct formal calculations and apply the principle of the conservation of the index of noether operator after it finite dimensional extension in the new constructed functional space. The noetherity of the extended operator is established and the deficient numbers with the corresponding index are calculated in various cases investigated.

Keywords: Integral Equation of Third Kind, Deficients Numbers, Noether Operator, Fundamental Functions, Singular Integral Operator

1. Introduction

The importance of the investigation of noetherity of some integro-differential operators defined by third kind integral equations is well known as researches undertook in such topic is widely illustrated through the works of many scientists [15, 16]. We recall that Rogozhin V. S., Raslambekov S. N., and Gobbassov N. S respectively in their works, had constructed noether theory for some integro-differential operators defined similarly but having some specificities on the considered integral equation. Among others, the question of the solvability of the considered integral equations, in such researches were investigated case by case in full details [10, 13-16].

By their side Bart G. R., and Warnock R. L have quite widely illustrated in their scientific researches, the theorems on the solvabilty conditions for some types of linear integral equations of third kind which are defined by integro-differential operators. From that, it is seen the noetherity nature of the considered linear operators [4, 5, 18, 19].

In the same direction, special approaches have been developed and applied under construction of noether theory for integro-differential operators defined by third kind integral equations in various specific functional and generalized spaces. Illustration of such approaches may be found in the following scientific researches and papers [6-11].

The idea develloped and used by Abdourahman. A, Karapetiants. N in their article «Noether theory for third kind linear integral equation with a singular linear differential operator in the main part» published in the Proceedings of A. Razmadze Math. Institute 135, 1-26 (2004), clearly presented the needed approach undertook when it has been necessary to introduce the concept and notion of Taylor derivatives, associated spaces and associated operators to reach the goal (guarateeing the noetherity of the studied operator) [12, 23-25].

We also note that, Bart G. R., Warnock R. L., Rogozhin V. S., Raslambekov S. N., and Gobbassov N. S., each of them had realized some cases of extension of their initial noether operator and therefore, established noetherity of the considered extended operator.

We recall that, related to our work, the noetherity of the initial operator considered in this work of the following form.

$$(A\varphi)(x) = x^p\varphi'(x) + \int_{-1}^{1} K(x,t)\,\varphi(t)dt = f(x); x \in [-1,1],$$

with $\varphi \in C_{-1}^{1}[-1,1]$, $f(x) \in C_0^{\{P\}}[-1,1]$ and $K(x,t) \in C_0^{\{P\}}[-1,1]\ \mathrm{X}\ \mathrm{C}\,[-1,1]$ is completely established with full details in [11, 12, 23].

E. Tompé Weimbapou, Abdourahman, and E. Kengne, in their recent article [21], have realized the extension of this mentioned upstair operator when the unknown function rather than $\varphi \in C_{-1}^{1}[-1,1]$ is taken as following $\varphi \in D_m = C_{-1}^{1}[-1,1] \oplus \left\{\sum_{k=0}^{m} \alpha_k \delta^{\{k\}}(x)\right\}$ with the condition $0 \le m < p-2$. The noetherity of the extended operator noted $\bar{A}$ was established and its index $\chi(\bar{A})$ is calculated.

Following such previous researches done in the mentionned upstair cases, we are focussing our present work into the realization of the extension of the initial operator in the case when $m > p-2$ and to establish the noetherity of $\bar{A}$.

Namely and here in this paper, we establish the extension of the following noether operator defined by the third kind singular integral equation.

$$(A\varphi)(x) = x^p\varphi'(x) + \int_{-1}^{1} K(x,t)\,\varphi(t)dt = f(x); x \in [-1,1], \tag{1}$$

where $\varphi \in C_{-1}^{1}[-1,1], f(x) \in C_0^{\{P\}}[-1,1]$ and $K(x,t) \in C_0^{\{P\}}[-1,1]\ \mathrm{X}\ \mathrm{C}\,[-1,1]$ with Dirac delta functions and it derivatives of some order.

e $\varphi \in D_m = C_{-1}^{1}[-1,1] \oplus \left\{\sum_{k=0}^{m} \alpha_k \delta^{\{k\}}(x)\right\};\ m > p-2.$

And next, we establish the noetherity of the extended operator.

This paper is structured as follow: in section 2, we recall some fundamental well known concepts of noether theory, third kind integro-differential operator, associated spaces, associated operators, and Taylor derivatives. Section 3 presenting the main results of the paper is devoted properly first of all to the realization of the extension of the operator *A* when taking the unknown function from the banach space D_m and secondly to the establishment of noetherity of the extended operator. We summarize our work and conclude in section 4, followed by some recommendations for the follow-up or future scientific works to undertake, stated in section 5.

2. Preliminaries

Before we present in details our main results, the following definitions and concepts well known from the noether theory of operators are required. Within this work, we also use the notions of a linear Fredholm integral equation of the third kind widely studied in many works [14, 15, 17, 20, 21]. We also underline that in transport theory, it is well indicated the natural space of solutions of a linear Fredholm third kind integral equation of the following form $\mathrm{A_n}\varphi = f$, where the right hand side function f is given as function of the variable $x \in [\mathrm{a,b}]$, and φ is the unknown function of the variable $x \in [\mathrm{a,b}]$. The space of generalized functions containing the solutions of such third kind integral equation is denoted by

$$C_{D_n}^{P_n} = \left\{\varphi(x)/\varphi(x) = y(x) + \sum_{k=1}^{n} \alpha_k \mathrm{P}\frac{1}{x-x_k} + \sum_{k=1}^{n} \omega_k \delta_k(x-x_k)\right\}$$

Details of such researches are found in [1-3, 21].

By the way, we briefly review this important notion of Taylor derivatives which is widely used when constructing noether theory of the considered operator *A* and it extension noted $\bar{A}$.

Definition 2.1 Function $\varphi(x) \in C[-1,1]$ admits at the point $x = 0$ Taylor derivative up to the order $p \in \mathbb{N}$ if there exists recurrently for $k = 1,2,\dots,p$, the following limits:

$$\varphi^{\{k\}}(0) = k!\, \lim_{x\to 0} x^{-k}\left[\varphi(x) - \sum_{j=0}^{k-1}\frac{\varphi^{\{j\}}(0)}{j!}x^j\right]. \tag{2}$$

The class of such functions $\varphi(x)$ is noted $C_0^{\{P\}}[-1,1]$.

We say that the kernel $k(x,t) \in C_0^{\{P\}}[-1,1]\ X C[-1,1]$, if and only if $k(x,t) \in \mathrm{C}[-1,1]\ \mathrm{X}\ \mathrm{C}[-1,1]$ and admits Taylor derivatives according to the variable *x* at the point (0,*t*) whatever $t \in [-1,1]$.

Let us move to the next following important concepts.

Associated operator and associated space.

Definition 2.2. The Banach space $\mathrm{E}' \subset \mathrm{E}^*$ is called associated space with the space E, if

$$|(\mathrm{f},\varphi)| \le \mathrm{c}\|\mathrm{f}\|_{\mathrm{E}'}\|\varphi\|_{\mathrm{E}} \tag{3}$$

for every $\varphi \in \mathrm{E}, \mathrm{f} \in \mathrm{E}'$.

We note that the initial space E can be considered associated with the space E′. Moreover, the norm $\|\mathrm{f}\|_{\mathrm{E}'}$ is not obliged to be equivalent to the norm $\|\mathrm{f}\|_{\mathrm{E}^*}$.

Let be noted $\mathcal{L}(\mathrm{E}_1,\mathrm{E}_2)$ the banach algebra of all linear bounded operators from E_1 into E_2.

Definition 2.3. Let $\mathrm{E_j}, \mathrm{j} = 1,2$ two banach spaces and $\mathrm{E'_j}$ their associated spaces. The operators $\mathrm{A} \in \mathcal{L}(\mathrm{E}_1,\mathrm{E}_2)$ and $\mathrm{A}' \in \mathcal{L}(\mathrm{E'}_2,\mathrm{E'}_1)$ are called associated, if

$$(\mathrm{A'f},\varphi) = (\mathrm{f},\mathrm{A}\varphi) \tag{4}$$

for all $f \in E'_2$ and $\varphi \in E_1$.

By defining the concept of associated space and associated operator, we used the work [22].

Further we will also use some concepts of noether theory from [22].

For the operator $A \in \mathcal{L}(E_1, E_2)$ we put $\alpha(A) = \dim\ker A$ − the number (of linearly independent) zero of the operator A; and $\beta(A) = \dim\operatorname{coker}A$ − the number of zero of the conjugate operator in the conjugate space; $\chi(A) = \alpha(A) - \beta(A)$ − the index of the operator.

In the case when $\alpha(A)$ and $\beta(A)$ are finite, and the image of the operator A closed in E_2, then the operator A is called noether operator.

It seems that, we can formalise the noetherity in terms of associated operator and associated space [22, 26].

Lemma 2.1 Let $E_j, j = 1,2$ two banach spaces and E'_j their associated spaces and, let $A \in \mathcal{L}(E_1, E_2)$ and $A' \in \mathcal{L}(E'_2, E'_1)$ be associated noether operators and more,

$$\alpha(A) = -\alpha(A').$$

Then, for the solvability of the equation $A\varphi = f$ it is necessary and sufficient that $(f, \psi) = 0$ for all solutions of the homogeneous associated equation $A'\psi = 0$.

Pair of associated spaces.

We give the following definition.

Definition 2.4 Let $x_0 \in [-1,1]$. Through $C^1_{x_0}[-1,1]$ we represente the set of functions from $C^1[-1,1]$ verifiying the condition $\varphi(x_0) = 0$.

It is clear that, $C^1_{x_0}[-1,1]$ is a banach subspace in the space $C^1[-1,1]$, if remarking, that for $\varphi_n(x_0) \in C^1_{x_0}[-1,1]$ the convergence by norm $C^1[-1,1]$ conducts $\varphi_n(x_0) \to \varphi(x_0), n \to \infty$, that, with respect to $\varphi_n(x_0) = 0$ for all $n \in \mathbb{N}$ leads us to $\varphi(x_0) = 0$.

Now let us state this important lemma.

Lemma 2.2 The space $C^1_{x_0}[-1,1]$ is associated to the space $C[-1,1]$.

Proof: It is sufficient to ensure that for the regular functional (f, φ), where $f \in C^1_{x_0}[-1,1]$ and $\varphi \in C[-1,1]$ it is taking place the approximation of the form (3):

$$|(f,\varphi)| \le c\|f\|_{C^1_{x_0}[-1,1]}\|\varphi\|_{C[-1,1]}$$

with some constant $c > 0$. The last is obvious as

$$|(f,\varphi)| = \left|\int_{-1}^{1} f(x)\varphi(x)dx\right| \le 2\max_{-1\le x\le 1}|f(x)| \cdot \max_{-1\le x\le 1}|\varphi(x)| \cdot \tag{5}$$

That is what was required.

Let make a remark. From the approximation (5) it can be seen that, the associated with the space $C[-1,1]$ should be the spaces $C^1[-1,1]$ and $C[-1,1]$, as in (5) it has not been used the approximation of the derivative and the value at the point x_0.

Therefore, we can, narrowing the associated space, pick up which one is convenient for our further goals. Namely for that reason in the lemma 2.2, it is featured the space $C^1_{x_0}[-1,1]$.

Definition 2.5 Through $P^1 = P^{1,\{p\}}_{1,0}[-1,1]$ we note the space of generalized functions $\psi(x)$ on the subspace of test functions $C_0^{\{P\}}[-1,1]$ such that,

$$\psi(x) = \frac{z(x)}{x^p} + \sum_{k=0}^{p-1}\beta_k\delta^{\{k\}}(x), \tag{6}$$

where $z(x) \in C_0^{\{P\}}[-1,1] \cap C^1_{-1}[-1,1]$, $\beta_k -$ arbitrary constants $\delta^{\{k\}}(x) -$ *k-th* Taylor derivative of Dirac delta function which can be understood in the following way.

$$(\delta^{\{k\}}(x), \varphi(x)) = \int_{-1}^{1}\delta^{\{k\}}(x)\,\varphi(x)dx = (-1)^k\varphi^{\{k\}}(0).$$

In the space P^1 let introduce the norm.

$$\|\psi\|_{P^1} = \|z\|_{C_0^{\{P\}}[-1,1]} + \|z\|_{C^1[-1,1]} + \sum_{k=0}^{p-1}|\beta_k|, \tag{7}$$

We note that under $p = 1$ the expression of the norm can be writing in a more simple way.

In fact, from the equality $z(x) = x^p(N^pz)(x) + \sum_{k=0}^{p-1}\frac{z^{\{k\}}(0)}{k!}x^k$ it follows, that (under $p \in \mathbb{N}$)

$$\|z(x)\|_{C[-1,1]} \le \|N^pz\|_{C[-1,1]} + \sum_{k=0}^{p-1}\left|z^{\{k\}}(0)\right| = \|z(x)\|_{C_0^{\{P\}}[-1,1]}.$$

In the case $p = 1$ we have:

$$\|z\|_{C_0^{\{1\}}[-1,1]} = \left\|\frac{z(x)-z(0)}{x}\right\|_{C[-1,1]} + |z(0)| \le \|z\|_{C^1[-1,1]},$$

so that under $p = 1$:

$$\|z\|_{C[-1,1]} \le \|z\|_{C_0^{\{1\}}[-1,1]} \le \|z\|_{C^1[-1,1]}. \tag{8}$$

From (7) it follows that under $p = 1$ the norm in (8) can be defined in the following way:

$$\|\psi\|_{P^1} = \|z\|_{C^1[-1,1]} + |\beta_0| \tag{9}$$

Theorem 2.1 The space P^1 is a banach space associated with $C_0^{\{P\}}[-1,1]$.

Proof: The fact that P^1 is a banach space follows from the definition (7) where the norm in P^1 is defined as sum of the norms in the banach spaces $C_0^{\{P\}}[-1,1]$ and $C^1[-1,1]$ with addition of a norm of finite-dimensional space.

The fact that $|(f,\psi)| \le c\|f\|_{C_0^{\{P\}}[-1,1]}\|\psi\|_{P^1}$ can be obtained analogously as done in [21].

Next, we move to the following section in which we present the main results of our work.

3. Main Results

Before we present the main results of the work in this section, let give supplementary important definitions.

Defintion 3.1 We note through $D_m = C^1_{-1}[-1,1] \oplus$

$\{\sum_{k=0}^{m}\alpha_k\delta^{\{k\}}(x)\}$ the space of all functions $\varphi(x)$ presented as following:

$$\varphi(x)=\varphi_0(x)+\sum_{k=0}^{m}\alpha_k\delta^{\{k\}}(x) \tag{10}$$

where $\varphi_0(x)\in C^1[-1,1]$ and $\varphi_0(-1)=0$ with the natural norm:

$$\|\varphi(x)\|_{D_m}=\|\varphi_0(x)\|_{C[-1,1]}+\sum_{k=1}^{m}|\alpha_k|. \tag{11}$$

Defintion 3.2 Let $l\geq p-1, p\in\mathbb{N}$. Through $C_{0,l}^{\{P\}}[-1,1]$, we represente the space of all functions f(x), having the folllowing form:

$$\mathrm{f(x)}=\mathrm{f}_0(\mathrm{x})+\sum_{k=0}^{l-p+1}\beta_k\delta^{\{k\}}(x),$$

where $\mathrm{f}_0(\mathrm{x})\in C_0^{\{P\}}[-1,1]$.

The space $C_{0,l}^{\{P\}}[-1,1]$ becomes a banach space if we introduce the normin the following way:

$$\|f\|_{C_{0,l}^{\{P\}}[-1,1]}=\|\mathrm{f}_0\|_{C_0^{\{P\}}[-1,1]}+\sum_{k=0}^{l-p+1}|\beta_k| \tag{12}$$

We note through $\bar{A}$ the extension of the operator A and in the case to be investigated and we consider as previously said, the case when $m>p-2$. So that we study the situation when the operator $\bar{A}$ is investigated from D_m into $C_{0,l}^{\{P\}}[-1,1]$ under $l=m$.

For this goal, we will consider first of all, the main part of the operator $\bar{A}$ noted $\bar{L}=x^p\varphi'(x)$. It is clear that the number of the solutions of the homogenous equation is equal to $p-1$ and this is not depending of the relationship between m and p for all $m\geq p-2$.

Concerning the nonhomogeneous equation then we have,

$$\bar{L}\varphi=x^p\varphi'(x)=x^p\varphi'_0(x)+\sum_{k=p-1}^{m}\alpha_k x^p\delta^{\{k+1\}}(x)=x^p\varphi'_0(x)+\sum_{k=p-1}^{m}\alpha_k\frac{(-1)^p k!}{(k-p)!}\delta^{\{k-p+1\}}(x)=f(x) \tag{13}$$

Taking into account the representation for the function $f(x)\in C_{0,m}^{\{P\}}[-1,1]$ in the form of

$$\mathrm{f(x)}=\mathrm{f}_0(\mathrm{x})+\sum_{k=0}^{l-p+1}\beta_k\delta^{\{k\}}(x)$$

We see that (14) naturally breaks down into two equations.

$$x^p\varphi'_0(x)=\mathrm{f}_0(\mathrm{x}) \tag{14}$$

$$\sum_{k=0}^{m-p+1}\alpha_{k+p-1}\frac{(-1)^p(k+p-1)!}{(k-1)!}\delta^{\{k\}}(x)=\sum_{k=0}^{m-p+1}\beta_k\,\delta^{\{k\}}(x) \tag{15}$$

We see that the coefficients α_k are defined uniquelly by the formulas.

$$\alpha_k=(-1)^p\frac{(k-p)!}{k!}\beta_{k-p+1}, k=p-1,\ldots.,m. \tag{16}$$

Under accomplishment of p $-$ conditions:

$$\mathrm{f}_0(0)=f_0^{\{1\}}(0)=\ldots\ldots=f_0^{\{p-1\}}(0),$$

the solution of the nonhomogeneous equation is given by the formula:

$$\varphi(x)=\int_{-1}^{x}\frac{\mathrm{f}_0(\mathrm{t})}{t^p}dt+\sum_{k=p-1}^{m}\frac{(-1)^p(k-p)!}{k!}\beta_{k-p+1}\delta^{\{k\}}(x)+\sum_{k=0}^{p-2}C_k\,\delta^{\{k\}}(x), \tag{17}$$

where C_k are arbitrary constants.

It is obvious that we have in this case $\alpha(\bar{L})=\mathrm{p}-1, \beta(\bar{L})=\mathrm{p}$ and $\chi(\bar{L})=-1$ so that it is conserved the result related to the conservation of the index of noether operator after realizing the extension [22].

So that it is holding place the following important theorem:

Theorem 3.1 The operator $\bar{A}$: $D_m\to C_{0,m}^{\{P\}}[-1,1]$ under $m\geq p-2$ is a noether operator with the index -1.

Naturally, we may have formulated theorem 3.1 with the use of the notion of associated operator but, however on this way, we meet the necessity to define the notion of associated operator somewhat formal.

Infact, if we hold formal calculations, then we have

$$(\bar{A}\varphi,\psi)=\left(x^p\varphi'(x)+\int_{-1}^{1}k(x,t)\varphi(t)\,dt,\frac{z(x)}{x^p}+\sum_{i=0}^{p-1}\omega_i\,\delta^{\{i\}}(x)\right)=$$

$$\left(x^p\varphi_0(x)+\sum_{k=0}^{m}\alpha_k\,x^p\delta^{\{k+1\}}(x)+\int_{-1}^{1}k(x,t)\varphi_0(t)\,dt+\sum_{k=0}^{m}\alpha_k\int_{-1}^{1}k(x,t)\delta^{\{k\}}(t)\,dt,\frac{z(x)}{x^p}+\sum_{i=0}^{p-1}\omega_i\,\delta^{\{i\}}(x)\right)=$$

$$\left(\varphi'_0(x),z(x)\right)+\sum_{k=p-1}^{m}(-1)^p\alpha_k\frac{(k+1)!}{(k+1-p)!}(\delta^{\{k+1-p\}}(x),\frac{z(x)}{x^p})+\sum_{k=p-1}^{m}(-1)^p\alpha_k\frac{(k+1)!}{(k+1-p)!}\sum_{i=0}^{p-1}\omega_i\,(\delta^{\{k+1-p\}}(x),\delta^{\{i\}}(x))+$$

$$(\int_{-1}^{1}k(x,t)\varphi_0(t)\,dt,\frac{z(x)}{x^p})+\sum_{i=0}^{p-1}(-1)^i\omega_i\int_{-1}^{1}k_1^{\{i\}}(0,t)\varphi_0(t)\,dt+\sum_{k=0}^{m}(-1)^k\alpha_k(\,k_2^{\{k\}}(x,0),\frac{z(x)}{x^p})+$$

$$\sum_{k=0}^{m}(-1)^k\alpha_k\sum_{i=0}^{p-1}(-1)^i\omega_i\,k_{21}^{\{k\}\{i\}}(0,0).$$

On the other side, let consider the following computation:

$$(\varphi, \bar{A}'\psi) = (\varphi_0(x) + \sum_{k=0}^{m} \alpha_k \delta^{\{k\}}(x), -(x^p\psi)' + \int_{-1}^{1} k(x,t)\psi(t)\,dt) =$$
$$(\varphi_0(x) + \sum_{k=0}^{m} \alpha_k \delta^{\{k\}}(x), -z'(x) + \int_{-1}^{1} k(t,x)\frac{z(t)}{t^p}dt + \sum_{i=0}^{p-1}(-1)^i k_1^{\{i\}}(0,x) = (\varphi_0(x), -z'(x)) +$$
$$(\varphi_0(x), \int_{-1}^{1} k(t,x)\frac{z(t)}{t^p}dt) +$$
$$\sum_{i=0}^{p-1}(-1)^i \omega_i(\varphi_0(x), k_1^{\{i\}}(0,x)) -$$
$$\sum_{k=0}^{m} \alpha_k(-1)^k z^{\{k+1\}}(0) + \sum_{k=0}^{m}(-1)^k \alpha_k \int_{-1}^{1} k_2^{\{k\}}(t,0)\frac{z(t)}{t^p}dt + \sum_{k=0}^{m}(-1)^k \alpha_k \sum_{i=0}^{p-1}(-1)^i \omega_i k_{12}^{\{i\}\{k\}}(0,0).$$

To get an alliance for the two operators and to be accomplished the equality $(\bar{A}\varphi, \psi) = (\varphi, \bar{A}'\psi)$, where $\varphi \in D_m$, $\psi \in \overline{\mathrm{P}^1}$ and $\overline{A} : D_m \to C_{0,m}^{\{P\}}[-1,1]$, $\bar{A}': \overline{\mathrm{P}^1} \to \mathrm{C}[-1,1]$, it is sufficient that it is realized and defined.

$$\left(\delta^{\{i\}}(x), \delta^{\{k\}}(x)\right) = 0;\ \forall\, \mathrm{k} \geq 0, \mathrm{i} \geq 0 \qquad (18)$$

and more over by the definition.

$$z^{\{k+1\}}(0) = (-1)^{k+1+p} \frac{(k+1)!}{(k+1-p)!}(\delta^{\{k+1-p\}}(x), \frac{z(x)}{x^p}). \quad (19)$$

We remark that the last is not playing the role of the condition and just serves only a formal definition of the right part appearing when computing the expression of the following form:

$$(x^p\delta^{\{k+1\}}(x), \frac{z(x)}{x^p}) = (-1)^{k+1}z^{\{k+1\}}(0). \qquad (20)$$

Under such supplementary suppositions, we may have considered the operators $\overline{A}$ and $\bar{A}'$ are associated operators. The structure of the space D_m is such that:

$$D_m = C_{-1}^{1}[-1,1] \oplus K_m,$$

where K_m —is a finite dimensional space. Therefore and for that:

$$A: C_{-1}^{1}[-1,1] \to C_0^{\{P\}}[-1,1]$$

and $A : K_m \to \widetilde{K_l}$, where $\widetilde{K_l}$ analogously space of delta functions and it derivatives. All these allows us (supposing and using (18)) to consider the space D_m and $\overline{\mathrm{P}^1}$ quasi - allied. See [23].

Integral equation in the case $m > p - 2$.

Before we state the properly concern results, let recall that, we have investigated the case $m = p - 2$ when considering the space D_m in [21]. If considering $0 \leq m < p - 2$ as done previously while constructing the associated operator, then one can reach to $(m + 1) -$ conditions: $z^{\{1\}}(0) = z^{\{2\}}(0) = \ldots = z^{\{m+1\}}(0) = 0$ which guarantees the noetherity of the considered operator. See also [21].

Previously also done, we have considered the operator $\overline{A}$ acting from D_m into $C_{0,l}^{\{P\}}[-1,1]$ under $l = m$ and $m \leq p - 2$. It is clear that from the condition $l = m$ it is easy to give up. If l and m are arbitrary then the system (14) and (15) is taking the following form:

$$x^p\varphi'_0(x) = \mathrm{f}_0(x) \qquad (21)$$

$$\sum_{k=p-1}^{m} \alpha_k \frac{(-1)^p k!}{(k-p)!}\delta^{\{k-p+1\}}(x) = \sum_{k=0}^{l-p+1} \beta_k\, \delta^{\{k\}}(x) \quad (22)$$

If $l > m$, then from (22) we obtain the necessity that

$$\beta_k = 0, k = m - p + 2, \ldots\ldots, l - p + 1, \qquad (23)$$

and the others α_k are defined unequivocally by the formula (16).

Therefore, in this case:

$$\alpha(\bar{L}) = \mathrm{p} - 1, \beta(\bar{L}) = \mathrm{p} + (l - m)$$

where $l - m$ conditions defined by (23) and $p$conditions on $f_0(x)$:

$$f_0(0) = f_0^{\{1\}}(0) = \ldots = f_0^{\{p-1\}}(0) = 0$$

and therefore, $\chi(\bar{L}) = -1 + l - m$.

In the case, when $l < m$ the operator $\overline{A}$ is not acting at all from D_m into $C_{0,l}^{\{P\}}[-1,1]$ and for this reason we do not consider this situation here.

Consequently, under $l > m$ it is taking place the following generalization of theorem 3.1.

Theorem 3.2. The operator $\overline{A}: D_m \to C_{0,l}^{\{P\}}[-1,1]$ when $l > m \geq p - 2$ is a noether operator with the index $\chi(\overline{A}) = -1 + l - m$.

The proof of this theorem is obvious and based on the fact that after realizing extension, the initial operator and the extended operator are keeping the same index [22].

4. Conclusion

Summarizing our work, we have realized the extension of a noether operator A defined by the extended operator $\overline{A}$ in the space D_m. The realization of the mentioned extension that we conducted in this work was done step by step, according to the models carried out in our previous work and also in the researches of the authors Raslambekov S. N, Gobassov N. S and Sukavanam N. The realizations of the extensions, depending on the case, are carried out by highlighting the conditions of noetherity of the extension of the initial operator A as done in [21]. We considered first of all the operator A as a sum of two operators L and K where L is the operator defined by $L\varphi = x^p\varphi'$ and K is a compact operator defined $K\varphi = \int_{-1}^{1} k(x,t)\varphi(t)dt$. The first result we obtained concerning associated operators $\overline{L}'$ of $\overline{L}$, permitted us to investigate the noetherity of operators $\overline{L}$. According to the noetherity of operator $\overline{L}$, we investigated the noetherity of operator $\overline{A}$and formulated the results in theorem 3.1, when

$m \geq p-2$. A general situation was investigated for the noetherity of the operator $\overline{A}$ when $l > m \geq p-2$, and theorem 3.2 contains the main results obtained. Indeed, it has been established, that once the extension of the initial operator A has been carried out in the functional spaces considered, under what conditions the new operator $\overline{A}$ resulting from the extension of A becomes noether. Thus, the deficient numbers in addition to the index of the operator are calculated and the principle of the conservation of the index of the operator after extension was respected.

5. Recommandations

On the basis of what has been done by many scientific researchers and also namely in this work, it would be very interesting and challenging to continue the investigation for noetherity question of the extended operator of the initial noether operator when at this time, we take the unknown generalized function from the space $V_m = C^1_{-1}[-1,1] \oplus \left\{\sum_{k=1}^{m} \alpha_k F.p\frac{1}{x^k}\right\}$. This will the next work to be done in a brief future.

The main difficulty appearing when realizing such extension is still connected with the derivative of the unknown function within the third kind singular integral equation throuhg which is defined the initial operator to be extended onto the new generalized functional space. The present work in this paper achieved and the upcoming new problem to be solved, can allow us to completely cover such topic in the future, when we will set the problem of the extension onto a more large generalized functional space of the following way:

$$T_m = C^1_{-1}[-1,1] \oplus \left\{\sum_{k=0}^{m} \alpha_k \delta^{\{k\}}(x)\right\} \oplus \left\{\sum_{k=1}^{m} \alpha_k F.p\frac{1}{x^k}\right\}$$

as done by many scientific researchers, namely cited Gabbasov N. S. Raslambekov S. N, Bart G. R. and Warnock R. L.

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