International Journal of Theoretical and Applied Mathematics
2022; 8(3): 58-64
http://www.sciencepublishinggroup.com/j/ijtam
doi: 10.11648/j.ijtam.20220803.12
ISSN: 2575-5072 (Print); ISSN: 2575-5080 (Online)

Hyers-Ulam Stability of First Order Nonlinear Delay Difference Equations

Arun Kumar Tripathy[1,*], **Binayak Dihudi**[1,2]

[1]Department of Mathematics, Sambalpur University, Sambalpur, India
[2]Department of Mathematics, Konark Institute of Science and Technology, Bhubaneswar, India

Email address:

arun_tripathy70@rediffmail.com (Arun Kumar Tripathy), bdihudi@gmail.com (Binayak Dihudi)
[*]Corresponding author

To cite this article:

Arun Kumar Tripathy, Binayak Dihudi. Hyers-Ulam Stability of First Order Nonlinear Delay Difference Equations. *International Journal of Theoretical and Applied Mathematics*. Vol. 8, No. 3, 2022, pp. 58-64. doi: 10.11648/j.ijtam.20220803.12

Received: June 23, 2022; **Accepted:** August 5, 2022; **Published:** September 28, 2022

Abstract: Initially Ulam's stability problem has originated for functional equations of both linear and nonlinear types. Because of Hyers and Rassias, the Ulam's stability problem has come to different shapes over different spaces. Slowly, it has got its name as Hyers-Ulam stability and Hyers-Ulam-Rassias stability. Meanwhile, the Hyers-Ulam stability has been extended to special functional equations like differential and difference equations. In this study, we examine the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of a class of first-order nonlinear delay difference equations with real coefficients on Banach space. Also, its nonhomogeneous counterpart has been studied for the same. Why we are interested in this study is that this is a special type of stability unlike the so called stability of differential or difference equations. As soon as we locate a solution in a Banach space, it is in the ϵ -neighbourhood while the concerned difference inequality is in an ϵ -neighbourhood. Lipschitz condition and Banach's fixed point theorem are our state of art to apply. Main results are illustrated by the examples.

Keywords: Hyers-Ulam Stability, Difference Equation, Nonlinear, Delay

1. Introduction

The study of stability problems for various functional equations originated from a talk given by S. M. Ulam in 1940 [23]. Ulam discussed a number of important unsolved problems duing that talk. An example of such a problem is a problem regarding the stability of functional equations and state that "Give conditions for a order mapping near approximately linear mapping to exist". In 1941, Hyers [3] gave an answer to the problem as follows:

Let E_1 and E_2 be two real Banach spaces and $u : E_1 \to E_2$ be a mapping. If there is an $\epsilon > 0$ such that

$$|u(x+y) - u(x) - u(y)| \leq \epsilon, \qquad (x, y \in E_1),$$

then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$|u(x) - T(x)| \leq \epsilon, \qquad x \in E_1.$$

Furthermore, the result of Hyers has been generalized by Rassias [18]. Many researchers have since extended Ulam's stability problems to other functional equations and generalised Hyer's result in a variety of ways (see for e.g., [4, 11, 12, 19]). In subsequent times, Ulam's stability problems for functional equations were replaced with stability problems for differential equations as well. The differential equation

$$a_n(\theta)y^{(n)}(\theta) + a_{n-1}(\theta)y^{(n-1)}(\theta) + \ldots + a_1(\theta)y'(\theta) + a_0(\theta)y(\theta) + h(\theta) = 0$$

has the Hyers-Ulam stability, if for given $\epsilon > 0$, I be an open interval and for any function u satisfying the differential inequality

$$|a_n(\theta)y^{(n)}(\theta)+a_{n-1}(\theta)y^{(n-1)}(\theta)+\ldots+a_1(\theta)y'(\theta)+a_0(\theta)y(\theta)+h(\theta)|\leq\epsilon,$$

Then there exists a solution $u_0(\theta)$ of the above equation such that $\mid u(\theta)-u_0(\theta)\mid\leq K(\epsilon)$ and $\lim_{\epsilon\to 0}K(\epsilon)=0, \theta\in I$.

If the preceding statement is also true when we replace ϵ and $K(\epsilon)$ by $\phi(t)$ and $\psi(t)$ respectively, where ϕ,ψ : $I\to[0,\infty)$ are functions independent of u and u_0 explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability. Keeping in view of the above fact, we formulate the following definition:

Definition 1.1. The difference equation

$$a_k(\xi)y(\xi+k)+a_{k-1}(\xi)y(\xi+k-1)+\ldots+a_1(\xi)y(\xi+1)+a_0(\xi)y(\xi)+h(\xi)=0$$

has the Hyers-Ulam stability, if for given $\epsilon>0$ and for any function f satisfying the inequality

$$\mid a_k(\xi)y(\xi+k)+a_{k-1}(\xi)y(\xi+k-1)+\ldots+a_1(\xi)y(\xi+1)+a_0(\xi)y(\xi)+h(\xi)\mid\leq\epsilon,$$

Then there exists a solution f_0 of the above difference equation such that $\mid f(\xi)-f_0(\xi)\mid\leq K(\epsilon)$ and $\lim_{\epsilon\to 0}K(\epsilon)=0$ for $\xi\in N(0)=\{0,1,2,3,\ldots\}$.

If we replace ϵ and $K(\epsilon)$ with $\varphi(\xi)$ and $\psi(\xi)$, where $\varphi(\xi)$ and $\psi(\xi)$ are real valued functions defined on $N(\xi_0)$, $\xi_0\geq 0$ for $\xi\in N(\xi_0)$, which are independent of f and f_0 explicitly, we say that the corresponding difference equation has the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability.

Several works on Hyers-Ulam stability have been done in the direction of differential equations. Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see for e.g., [15, 19]). After that, Alsina and Ger published their work [1], where they have proved the Hyers-Ulam stability of the differential equation $y'(t)=y(t)$. In this direction, we refer the works [5, 6, 7, 8, 10, 13, 14, 16, 20, 21] and the references cited therein.

The objective of this work is to investigate the Hyers-Ulam stability of the difference equations

$$a(\xi+1)y(\xi+1)-a(\xi)y(\xi)=b(\xi)g(y(\xi-\sigma(\xi))) \tag{1}$$

and

$$a(\xi+1)y(\xi+1)-a(\xi)y(\xi)=b(\xi)g(y(\xi-\sigma(\xi)))+h(\xi), \tag{2}$$

Where $a(\xi)$, $b(\xi)$, $\sigma(\xi)$ and $h(\xi)$ are real valued functions defined on $N(0)=\{0,1,2,3,\ldots\}$ such that $a,\sigma\in\mathbb{R}_+$, $b,h\in\mathbb{R}$, $\lim_{\xi\to\infty}(\xi-\sigma(\xi))=+\infty$, and $g\in C(\mathbb{R},\mathbb{R})$ such that $ug(u)>0$ for $u\neq 0$. It is worth mentioning that there has been little research on (1)/(2) with regard to Hyers-Ulam stability. Towards this end, we refer the reader to some works [2, 9, 17] of the literature.

Definition 1.2. By a solution of (1) or (2), we mean a real valued function $x(\xi)$ which satisfies (1) or (2) and such that

$$x(\xi)=\begin{cases}\eta(\xi), & \xi\in[-\xi_*,0]\\ y(\xi), & \xi\in N(0).\end{cases}$$

Definition 1.3. We say that (1) has the Hyers-Ulam-Rassias stability with respect to $\theta(\xi)$, if there exists $\epsilon>0$ with the following property: for each real valued function $x(\xi)$ satisfying

$$\mid a(\xi+1)y(\xi+1)-a(\xi)y(\xi)-b(\xi)g(y(\xi-\sigma(\xi)))\mid\leq\theta(\xi), \tag{3}$$

there exists a solution $x_0(\xi)$ of (1) such that

$$\mid x(\xi)-x_0(\xi)\mid\leq K\theta(\xi),\quad \xi\in N(0).$$

2. Hyers-Ulam Stabiity of (1) and (2)

This section is devoted for discussion on the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of (1) and (2).

Theorem 2.1. Let g be Lipschitzian on $[\xi^*,\infty)$ with Lipschitz constant L and (3) hold. Assume that

$$(A_1)\quad \limsup_{\xi\to\infty}\left(\frac{L}{a(\xi)}\sum_{s=0}^{\xi-1}\mid b(s)\mid\right)=\alpha<1,$$

$$(A_2)\quad \limsup_{\xi\to\infty}\left(\alpha+\frac{1}{a(\xi)}\right)=\beta<1$$

And

$$(A_3)\ \text{ there exists a real valued function } z(\xi)\ such\ that\ \frac{1}{a(\xi)}\sum_{s=0}^{\xi-1}\mid \theta(s)\mid\leq cz(\xi)\ for\ all\ \xi.$$

Then (1) has the Hyers-Ulam-Rassias stabiity.
Proof Let (A_1) and (A_2) be hold. Then there exist $\xi_1,\xi_2>0$ and $\beta_1,\beta_2>0$ such that

$$\frac{L}{a(\xi)}\sum_{s=0}^{\xi-1}\mid b(s)\mid)\leq\alpha_1,\xi\geq\xi_1$$

and

$$\alpha_1+\frac{1}{a(\xi)}\leq\beta_1,\xi\geq\xi_2.$$

Let $X=l^{\infty}_{\xi_3}$ be the Banach space of all the real valued functions $x(\xi)$ with the sup norm defined by

$$\|x\|=\sup_{\xi\geq 0}\mid x(\xi)\mid$$

for $\xi>\xi_3>\max\{\xi_1,\xi_2\}$.
Define a subset S of X as follows:

$$S=\{u(\xi):\|u\|<\delta,u(\xi)=\psi(\xi),\xi\in[-\xi_3^*,0]\}.$$

Clearly, S is a closed convex subset of X, and we define $T:S\to S$ by

$$Tu(\xi)=\begin{cases}\psi(\xi),\xi\in[-\xi_3^*,0],\\ \frac{\psi(0)}{a(\xi)}+\frac{1}{a(\xi)}\sum\limits_{s=0}^{\xi-1}b(s)g(u(s-\sigma(s))),\xi\geq 0.\end{cases}$$

Indeed,

$$\begin{aligned}\mid Tu(\xi)\mid &\leq\frac{\mid\psi(0)\mid}{\mid a(\xi)\mid}+\frac{1}{a(\xi)}\sum_{s=0}^{\xi-1}\mid b(s)\mid\mid g(u(s-\sigma(s))\mid\leq\frac{\delta}{a(\xi)}+\frac{L}{a(\xi)}\sum_{s=0}^{\xi-1}\mid b(s)\mid\mid(u(s-\sigma(s))\mid\\ &\leq\frac{\delta}{a(\xi)}+\frac{L\|u\|}{a(\xi)}\sum_{s=0}^{\xi-1}\mid b(s)\mid\leq\left(\alpha_1+\frac{1}{a(\xi)}\right)\delta\leq\beta_1\delta<\delta\end{aligned}$$

implies that $\|Tu\|<\delta$ and hence $T:S\to S$. For $u,v\in S$, we have

$$\begin{aligned}&\mid Tu(\xi)-Tv(\xi)\mid\leq\frac{1}{a(\xi)}\sum_{s=0}^{\xi-1}\mid b(s)\mid\mid g(u(s-\sigma(s)))-g(v(s-\sigma(s)))\mid\\ &\leq\frac{L}{a(\xi)}\sum_{s=0}^{\xi-1}\mid b(s)\mid\mid u(s-\sigma(s))-v(s-\sigma(s))\mid\leq\frac{L\|u-v\|}{a(\xi)}\sum_{s=0}^{\xi-1}\mid b(s)\mid\leq\alpha_1\|u-v\|\end{aligned}$$

implies that

$$\|Tu - Tv\| \leq \alpha_1 \|u - v\|.$$

Therefore, T is a contraction with $\alpha_1 < 1$. Hence by Banach's fixed point theorem, T has a unique fixed point and let it be $x_0(\xi) \in S$ for $\xi \in [-\xi_3^*, \infty)$. Thus,

$$x_0(\xi) = \begin{cases} \psi(\xi), \ \xi \in [-\xi_3^*, 0] \\ \frac{\psi(0)}{a(\xi)} + \frac{1}{a(\xi)} \sum\limits_{s=0}^{\xi-1} b(s) g(x_0(s - \sigma(s))), \ \xi \geq 0. \end{cases}$$

We may note that $\psi(0) = a(0)x(0)$. Form (3), it follows that

$$-\theta(\xi) \leq \Delta(a(\xi)x(\xi)) - b(\xi)g(x(\xi - \sigma(\xi))) \leq \theta(\xi)$$

for $\xi \in [-\xi_3^*, \infty)$. Therefore for $\xi \geq -\xi_3^*$,

$$-\sum_{s=0}^{\xi-1} \theta(s) \leq a(\xi)x(\xi) - a(0)x(0) - \sum_{s=0}^{\xi-1} b(s)g(x(s - \sigma(s))) \leq \sum_{s=0}^{\xi-1} \theta(s),$$

that is,

$$-\sum_{s=0}^{\xi-1} \theta(s) \leq x(\xi) - \frac{\psi(0)}{a(\xi)} - \frac{1}{a(\xi)} \sum_{s=0}^{\xi-1} b(s)(x(s - \sigma(s))) \leq \frac{1}{a(\xi)} \sum_{s=0}^{\xi-1} \theta(s).$$

Consequently,

$$\mid Tx(\xi) - x(\xi) \mid \leq cz(\xi), \ \xi \geq -\xi_3^*$$

due to (A_3). Now

$$\mid x(\xi) - x_0(\xi) \mid = \mid x(\xi) - Tx(\xi) + Tx(\xi) - x_0(\xi) \mid \leq \mid Tx(\xi) - x(\xi) \mid + \mid Tx(\xi) - Tx_0(\xi) \mid \leq cz(\xi) + \alpha_1 \mid x(\xi) - x_0(\xi) \mid$$

implies that

$$\mid x(\xi) - x_0(\xi) \mid \leq \frac{c}{1 - \alpha_1} z(\xi) = Kz(\xi), \xi \geq -\xi_3^*,$$

that is, (1) has the Hyers-Ulam-Rassias stability with the stability constant $K = \frac{c}{1-\alpha_1} > 0$. This completes the proof of the theorem.

Corollary 2.1. Let g be Lipschitzian on $[-\xi^*, \infty)$. Assume that (A_1), (A_2) and

$$(A_4) \quad \lim_{\xi \to \infty} \left(\frac{\xi}{a(\xi)} \right) = K, K > 0$$

Hold. If for $\epsilon > 0$

$$\mid \Delta(a(\xi)x(\xi) - b(\xi)g(x(\xi - \sigma(\xi))) \mid \leq \epsilon,$$

then (1) has the Hyers-Ulam Stability with the stability constant K.

Proof The proof of the corollary follows from the proof of Theorem 2.1. Hence the details are omitted.

Example 2.1. Consider

$$\Delta((\xi + 1)x(\xi)) + \frac{1}{4}(1 + \frac{1}{\xi + 1})x(\frac{\xi}{2}) = 0, \xi \geq 0, \tag{4}$$

where $\sigma(\xi) = \frac{\xi}{2}$, $a(\xi) = \xi + 1, b(\xi) = -\frac{1}{4}(1 + \frac{1}{\xi+1})$ and $g(x) = x$. Clearly $L = 1$ and

$$\limsup_{\xi \to \infty} \frac{L}{a(\xi)} \sum_{s=0}^{\xi-1} \mid b(s) \mid = \limsup_{\xi \to \infty} \frac{1}{4(\xi + 1)} \sum_{s=0}^{\xi-1} \left(1 + \frac{1}{\xi + 1} \right) \leq \limsup_{\xi \to \infty} \frac{2\xi}{4(\xi + 1)} = \frac{1}{2} = \alpha$$

implies that

$$\limsup_{\xi\to\infty}\left(\alpha+\frac{1}{\xi-1}\right)\leq\frac{1}{2}+\limsup_{\xi\to\infty}\left(\frac{1}{\xi+1}\right)=\frac{1}{2}=\beta.$$

Also,

$$\limsup_{\xi\to\infty}\left(\frac{\xi}{\xi+1}\right)=1=K.$$

If we choose $x(\xi)=\frac{1}{(\xi+1)^2}$, then $\psi(0)=1$ and

$$x_0(\xi)=\frac{1}{\xi+1}-\frac{1}{\xi+1}\sum_{s=0}^{\xi-1}(1+\frac{1}{(s+1)(s+2)}).$$

It is easy to verify that $\mid x(\xi)-x_0(\xi)\mid<\epsilon$ for $\xi\geq 0$. Hence by Corollary 2.1, (4) has the Hyers-Ulam stability with the stability constant $K=1$.

Theorem 2.2. Assume that

$$\mid a(\xi+1)a(\xi+1)-a(\xi)x(\xi)-b(\xi)g(x(\xi-\sigma(\xi))-h(\xi)\mid\leq\theta(\xi).$$

Let g be Lipschitzian on $[-\xi^*,\infty)$, with the Lipschitz constant L. If (A_3),

$$(A_5)\quad \limsup_{\xi\to\infty}(\frac{1}{a(\xi)})[L\sum_{s=0}^{\xi-1}\mid b(s)\mid+\frac{1}{\delta}\sum_{s=0}^{\xi-1}\mid h(s)\mid]=\nu<1,\delta>0\ is\ a\ constant$$

And

$$(A_6)\limsup_{\xi\to\infty}[\nu+\frac{1}{a(\xi)}]=\tau<1$$

Hold, then (2) is stable in the sense of Hyers-Ulam-Rassias stability.

Proof The first half of the proof of the theorem follows of Theorem 2.1. In this case,

$$Tu(\xi)=\begin{cases}\psi(\xi),\ \xi\in\ [-\xi_3^*,0]\\ \frac{\psi(0)}{a(\xi)}+\frac{1}{a(\xi)}\sum\limits_{s=0}^{\xi-1}[b(s)g(u(s-\sigma(s)))+h(s)],\ \xi\geq 0.\end{cases}$$

For the second half, we have that

$$-\theta(\xi)\leq\Delta(a(\xi)x(\xi))-b(\xi)g(x(\xi-\sigma(\xi)))-h(\xi)\leq\theta(\xi)$$

for $\xi\in[-\xi_3^*,\infty)$. For $\xi\geq-\xi_3^*$ and therefore,

$$-\sum_{s=0}^{\xi-1}\theta(s)\leq a(\xi)x(\xi)-a(0)x(0)-\sum_{s=0}^{\xi-1}[b(s)g(x(s-\sigma(s)))-h(s)]\leq\sum_{s=0}^{\xi-1}\theta(s),$$

that is,

$$-\frac{1}{a(\xi)}\sum_{s=0}^{\xi-1}\theta(s)\leq x(\xi)-\frac{\psi(0)}{a(\xi)}-\frac{1}{a(\xi)}\sum_{s=0}^{\xi-1}[b(s)g(x(s-\sigma(s)))-h(s)]\leq\frac{1}{a(\xi)}\sum_{s=0}^{\xi-1}\theta(s).$$

Consequently,

$$\mid Tx(\xi)-x(\xi)\mid\leq cz(\xi),\xi\geq-\xi_3^*$$

due to (A_3). The rest of the proof follows from the proof of the Theorem 2.1. Hence the theorem is proved.

Corollary 2.2. Let g be Lipschitzian on $[-\xi^*, \infty)$. Assume that $(A_4), (A_5)$ and (A_6) hold. If for $\epsilon > 0$,

$$| \Delta(a(\xi)x(\xi)) - b(\xi)g(x(\xi - \sigma(\xi))) - h(\xi) | \leq \epsilon,$$

then (2) has the Hyers-Ulam stability with the stability constant K.

Proof The proof of the theorem can be followed from Theorem 2.2. Hence the details are omitted.

Example 2.2. Consider

$$\Delta((\xi+1)x(\xi)) + \frac{1}{4}(1 + \frac{2}{\xi+1})x(\frac{\xi}{2}) = \frac{1}{(\xi+1)(\xi+2)^2} \tag{5}$$

for $\xi \geq 0$, where $\sigma(\xi) = \frac{\xi}{2}, a(\xi) = \xi + 1, b(\xi) = -\frac{1}{4}(1 + \frac{2}{\xi+1}), g(x) = x$ and $h(\xi) = \frac{1}{(\xi+1)(\xi+2)^2}$. Clearly, $L = 1$ and

$$\begin{aligned}
&\limsup_{\xi\to\infty} \frac{1}{a(\xi)}[L \sum_{s=0}^{\xi-1} | b(s) | + \frac{1}{\delta} \sum_{s=0}^{\xi-1} | h(s) |] \\
&\leq \limsup_{\xi\to\infty} \frac{1}{4(\xi+1)} \sum_{s=0}^{\xi-1} (1 + \frac{2}{s+1}) + \frac{1}{\delta} \limsup_{\xi\to\infty} \frac{1}{\xi+1} \sum_{s=0}^{\xi-1} \frac{1}{(s+1)(s+2)^2} \\
&\leq \limsup_{\xi\to\infty} \frac{1}{4(\xi+1)} \sum_{s=0}^{\xi-1} (1+2) + \frac{1}{\delta} \limsup_{\xi\to\infty} \frac{1}{\xi+1} \sum_{s=0}^{\xi-1} \frac{1}{(s+1)(s+2)} \\
&\leq \limsup_{\xi\to\infty} \frac{3\xi}{4(\xi+1)} + \frac{1}{\delta} \limsup_{\xi\to\infty} \frac{1}{\xi+1} \frac{\xi}{(\xi+1)} = \frac{3}{4} = \nu
\end{aligned}$$

implies that

$$\limsup_{\xi\to\infty} [\nu + \frac{1}{a(\xi)}] = \limsup_{\xi\to\infty} [\frac{3}{4} + \frac{1}{\xi+1}] \leq \frac{3}{4} = \tau.$$

Also, $\limsup_{\xi\to\infty} (\frac{\xi}{a(\xi)}) = 1 = K$. If we choose $x(\xi) = \frac{1}{(\xi+1)^2}$, then $\psi(0) = 1$ and $x_0(\xi) = \frac{1}{\xi+1} - \frac{\xi}{(\xi+1)^2}$.

It is easy to verify that $| x(\xi) - x_0(\xi) | \leq \epsilon$ for $\xi \geq 0$. Hence by Corollary 2.2, (5) has the Hyers-Ulam stability with the stability constant $K = 1$.

3. Discussion and Conclusion

The present work deals with the Hyers-Ulam stability properties of the nonlinear difference equations (1) and (2). In the light of the preceding method, it would be interesting to investigate the Hyers-Ulam stability properties of the second order nonlinear difference equations of the form:

$$a(\xi+2)y(\xi+2) - a(\xi+1)y(\xi+1) + a(\xi)y(\xi) = b(\xi)g(y(\xi - \sigma(\xi))) \tag{6}$$

and

$$a(\xi+2)y(\xi+2) - a(\xi+1)y(\xi+1) + a(\xi)y(\xi) = b(\xi)g(y(\xi - \sigma(\xi))) + h(\xi). \tag{7}$$

If $b(\xi) \equiv 0$, then (6) and (7) have been discussed for Hyers-Ulam stability in [22]. Otherwise, further discussion is left to the interested readers working in this area.

References

[1] C. Alsina, R. Ger (1998). On some inequalities and stability results related to the exponential function. J. Inequ. Appl. 2: 373-380.

[2] J. Brzdek, D. Popa, B. Xu (2010). Remarks on stability of linear recurrence of higher order. Appl. Math. Lett. 23: 1459-1463.

[3] D. H. Hyers (1941). On the stability of the linear functional equations. Proc. Natl. Acad. Sci. 27: 222-224.

[4] K. W. Jun, Y. H. Lee(1999). A generalization of the Hyers-Ulam-Rassias stability of the Jenson's equation. J. Math.Anal. Appl. 238:305-315.

[5] S. M. Jung (2004). Hyers-Ulam stability of linear differential equations of first order. Appl. Math. Lett. 17: 1135-1140.

[6] S. M. Jung (2005). Hyers-Ulam stability of linear differential equations of first order (III). J. Math. Anal. Appl. 311: 139-146.

[7] S. M. Jung (2006). Hyers-Ulam stability of linear differential equations of first order (II). Appl. Math. Lett. 19: 854-858.

[8] S. M. Jung, Th. M. Rassias; Ulam's problem for approximate homomorphisms in connection with Bernoulli's differential equation, Appl. Math. Comp. 187: 223-227.

[9] S. M. Jung (2015). Hyers -Ulam stability of the first order matrix difference equations, Adv. Differ. Equs. 2015: 1-13.

[10] Y. Li (2010). Hyers-Ulam stability of linear differential equations $y'' = \lambda^2 y$. Thai J. Math. 8: 215-219.

[11] T. Miura, S. E. Takahasi, H. Choda (2001). On the Hyers-Ulam stability of real continuous function valued differentiable map. Tokyo J. Math. 467-476.

[12] T. Miura (2002). On the Hyers -Ulam stability of a differentiable map. Sci. Math. Japan. 55: 17-24.

[13] T. Miura, S. Miyajima, S. E. Takahasi (2003). A characterization of Hyers-Ulam stability of first order linear differential operators J. Math. Anal. Appl. 286: 136-146.

[14] T. Miura, S. M. Jung, S. E. Takahasi (2004). Hyers-Ulam stability of the banach space valued linear differential equations $y' = \lambda y$. Korean Math. Soc. 41: 995-1005.

[15] M. Obloza (1993). Hyers stability of the linear differential equation. Rocznik Nauk-Dydakt. Proce. Mat. 13: 259-270.

[16] M. Obloza (1997). Connections between Hyers and Lyapunov stability of the ordinary differential equations. Rocznik Nauk-Dydakt. Proce. Mat. 14: 141-146.

[17] D. Popa (2005). Hyers-Ulam-Rassias stability of a linear recurrence. J. Math. Anal. Appl. 309: 591-597.

[18] Th. M. Rassias (1978). On the stability of linear mapping in Banach spaces. Proc. Amer. Math. Soc. 72: 297-300.

[19] Th. M. Rassias (2000). On the stability of functional equations and a problem of Ulam. Acta Appl. Math. 62: 23-30.

[20] S. E. Takahasi, H. Takagi, T. Miura, S.Miyajima (2004). The Hyers-Ulam stability constants of first order linear differential operators J. Math. Anal. Appl. 296: 403-409.

[21] A. K. Tripathy, A. Satapathy (2015). Hyers-Ulam stability of fourth order Euler's differential equations. J. Comp. Sci. Appl. Math. 1: 49-58.

[22] A. K. Tripathy (2021). Hyers-Ulam Stability of Ordinary Differential Equations. CRC Press.

[23] S. M. Ulam (1964). Problems in Modern Mathematics . Chapter VI. Wiley. New York.