International Journal of Theoretical and Applied Mathematics
2022; 8(4): 78-84
http://www.sciencepublishinggroup.com/j/ijtam
doi: 10.11648/j.ijtam.20220804.11
ISSN: 2575-5072 (Print); ISSN: 2575-5080 (Online)

A Generalized Gamma-Type Functions Involving Confluent Hypergeometric Mittage-Leffler Function and Associated Probability Distributions

Naresh Dudi

Department of Mathematics and Statistics, Jai Narain Vyas University, Jodhur, India

Email address:
nareshdudi@yahoo.co.in

To cite this article:
Naresh Dudi. A Generalized Gamma-Type Functions Involving Confluent Hypergeometric Mittage-Leffler Function and Associated Probability Distributions. *International Journal of Theoretical and Applied Mathematics*. Vol. 8, No. 4, 2022, pp. 78-84.
doi: 10.11648/j.ijtam.20220804.11

Received: May 23, 2022; **Accepted:** June 20, 2022; **Published:** October 21, 2022

Abstract: Gamma function is one of the important special functions occurring in many branches of mathematical physics and is investigated in detail in a number of literatures in recent years. The main object of this paper is to present a systematic study of a unification and generalization of the gamma-type functions, which is defined here by means of the confluent hypergeometric Mittage- Leffler function. Motivated essentially by the sources of the applications of the Mittage-Leffler functions in many areas of science and engineering, the author present in a unified manner. During the last two decades Mittage – Leffler function has come into prominence after about nine decades of its discovery by a Swedish Mathematician Mittage-Leffler, due to the vast potential of this applications in solving the problems of physical, biological, engineering, and earth sciences. In this article we investigated a probability density function associated with the generalized gamma-type function, together with several other related results in the theory of probability and statistics, are also considered.

Keywords: Special Function, Gamma and Incomplete Gamma Functions, Confluent Hypergeometric Mittage-Leffler Function, Probability Density Function, Moment Generating Function

1. Introduction

In this article we create a novel link between generalized gamma-type function and the confluent hypergeometric Mittage-Lefller functions of one, two, three and four parameters in the form in the form.

$$S^*\left(\begin{matrix}\lambda, a, b; c, d; \alpha, \beta; \gamma, \delta; \\ u, v\end{matrix} \quad p, k\right) := v^{-\lambda} \int_0^\infty t^{u-1} M_{\alpha,\gamma}^{\beta,\delta}(-pt) 3\mathrm{R}2(\lambda, a, b; c, d; k - \tfrac{t}{v}) dt, \tag{1}$$

where $M_{\alpha,\gamma}^{\beta,\delta}(z)$ is the confluent hypergeometric Mittage-Leffler function and a new probability density function involving generalized gamma function associated with the function ${}_3R_2^k(z) = {}_3\mathrm{R}_2(\lambda,a,b;c,d;k;z)$ which has been defined and studied by Saxena, Ram and Naresh [1]. This generalization provides unification and extension of the various generalization given earlier by Kobayashi [2, 3], Al-Musallum, Kalla [4, 5] and Virchenko et al. [6, 7].

2. Generalized Gamma Function

The present paper deals with a generalization of the gamma–type function associated with Confluent Hypergeometric Mittage-Leffler Function in the form.

$$S^*\left(\begin{matrix}\lambda, a, b; c, d; \alpha, \beta; \gamma, \delta; \\ u, v\end{matrix} \quad p, k\right) = v^{-\lambda} \int_0^\infty t^{u-1} M_{\alpha,\gamma}^{\beta,\delta}(-pt) 3\mathrm{R}2(\lambda, a, b; c, d; k - \tfrac{t}{v}) dt, \tag{2}$$

where $\lambda, \alpha, \beta, \gamma, \delta \in \mathbb{C}, Re(u) > 0, Re(p) > 0, k > 0, |argv| < \pi$.

The Mittage-Leffler function has earned the title of "Queen function of fractional calculus" for the fundamental role it plays within this subject. Certainly, this function is used to study of fractional integrals and derivatives; see, for example, Ghanim and Al-Janaby [8, 9], Ghanim et al. [10], Oros [11, 12], Haubold et al. [13], Paneva-Konovska [14], Mainardi and Gorenflo [15], Mathi and Haubold [16], Srivastava [17, 18], and Srivastava et al [19]. In 1903, Magnus Gustaf (gösta) Mittage-Leffler (1846-1927) [20] (also see Mittage-Leffler [21]), a Swedish mathematician, invented and studied the well-known Mittage-Leffler function $E_\alpha(z)$ given by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}\ (z \in \mathbb{C};\ \Re(\alpha) > 0). \quad (3)$$

Wiman [22] and References [23] then proposed a generalization $E_\alpha(z)$ of $E_{\alpha,\gamma}(z)$ provide by

$$E_{\alpha,\gamma}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\gamma)}\ (\Re(\alpha) > 0; \alpha, \gamma > \mathbb{C}). \quad (4)$$

As a result, the function of Mittage-Leffler $E_\alpha(z)$ and $E_{\alpha,\gamma}(z)$ in (3) and (4), respectively, have been studied and expended in variety of different ways and applications. The implementation in the physical model has succeeded in recent decades, the generalized Mittage-Leffler functions were also used in mathematical and physical issues, as the solutions of the fractional integral and differential equations were naturally presented. Ref. [24, 25] are two studies which have utilized the function (3) with parameters α and γ in more general functions linked to one or more parameters.

$$E_{\alpha,\gamma}^{\delta}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\delta)\, z^n}{\Gamma(\delta)\,\Gamma(\alpha n+\gamma)\, n!}, (\Re(\alpha) > 0; \alpha, \gamma, \delta > \mathbb{C}) \ (5)$$

$$E_{\alpha,\gamma}^{\beta,\delta}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\beta n+\delta)\, z^n}{\Gamma(\delta)\,\Gamma(\alpha n+\gamma)\, n!}, (\Re(\alpha) > 0; \alpha, \gamma, \delta > \mathbb{C}) \ (6)$$

The confluent hypergeometric Mittage-Leffler function $M_{\alpha,\gamma}^{\beta,\delta}(z)$ is given by E. Ghanim, Hiba F. Al-Janaby, Omar Bazighifan [26] in the form.

$$M_{\alpha,\gamma}^{\beta,\delta}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\gamma)\Gamma(\beta n+\delta)\, z^n}{\Gamma(\delta)\,\Gamma(\alpha n+\gamma)\, n!}, (\Re(\alpha) > 0; \alpha, \gamma, \beta, \delta > \mathbb{C}) \ (7)$$

The following are some special instances of the confluent hypergeometric Mittage-Leffler function $M_{\alpha,\gamma}^{\beta,\delta}(z)$:

$$M_{0,\gamma}^{0,\gamma}(z) = e^z, M_{0,1}^{1,1}(z) = \frac{1}{1-z}, M_{1,2}^{1,1}(z) = \frac{e^z-1}{z},$$

$$M_{2,1}^{1,1}(z^2) = cohz, M_{2,1}^{1,1}(-z^2) = cosz, M_{1,\gamma}^{1,\delta}(z) = M(\delta;\gamma;z),$$

$$M_{\alpha,1}^{1,1}(z) = E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}$$

where α is a positive integer, such that as n, then:

$$M_{\alpha,\gamma}^{1,\delta}(z) = 1\mathrm{Fn}\left(\delta; \frac{\gamma}{n}, \frac{\gamma+1}{n}, \dots, \frac{\gamma+n-1}{n}; \frac{z}{n^n}\right);$$

In addition

$$H_i(x,n) = x^{i-1} M_{n,i}^{1,1}(x^n), T_i(x,n) = x^{i-1} M_{n,i}^{1,1}(-x^n).$$

H_i and T_i are generalized hyperbolic and trigonometric functions, respectively [27].

3. Special Cases of Generalized Gamma Function

Generalized gamma -type function is of the form.

$$S^*\begin{pmatrix} \lambda, a, b; c, d; \alpha, \beta; \gamma, \delta; & p, k \\ u, v & \end{pmatrix} := v^{-\lambda} \int_0^\infty t^{u-1} M_{\alpha,\gamma}^{\beta,\delta}(-pt)\, 3\mathrm{R}2(\lambda, a, b; c, d; k - \tfrac{t}{v}) dt, \quad (8)$$

Case (i) For $\alpha = \beta = 0$ *and* $\gamma = \delta$ the confluent hypergeometric Mittage-Leffler function reduces to an exponential function e^{-pt} and equation (8) reduces to the following result given earlier by Chena Ram, Naresh [30].

$$S^*\begin{pmatrix} \lambda, a, b; c, d; 0,0; \gamma, \gamma; & p, k \\ u, v & \end{pmatrix} := v^{-\lambda} \int_0^\infty t^{u-1} e^{-pt}\, 3\mathrm{R}2(\lambda, a, b; c, d; k - \tfrac{t}{v}) dt, \quad (9)$$

where $Re(u) > 0, Re(p) > 0, k > 0, |argv| < \pi$.

Case (ii) For $\alpha = \beta = 0, \gamma = \delta$ and b=d, (8) reduces to the generalized gamma function discussed by Virchenko et al. [6].

$$S^*\begin{pmatrix} \lambda, a, b; c, b; 0,0; \gamma, \gamma; & p, k \\ u, v & \end{pmatrix} := v^{-\lambda} \int_0^\infty t^{u-1} e^{-pt}\, 2\mathrm{R}1(\lambda, a; c; k - \tfrac{t}{v}) dt, \quad (10)$$

where $Re(u) > 0, Re(p) > 0$.

Case (iii) For $\alpha = \beta = 1$ and b=d, equation (8) yields the following generalized gamma function studied by Saxena, Chena Ram, Naresh and Kalla [29].

$$S^*\begin{pmatrix} \lambda, a, b; c, b; 1,1; \gamma, \delta; & p, k \\ u, v & \end{pmatrix} := v^{-\lambda} \int_0^\infty t^{u-1}\, 1\phi1(\gamma, \delta; -pt)\, 2\mathrm{R}1(\lambda, a; c; k - \tfrac{t}{v}) dt, \quad (11)$$

where $Re(u) > 0, Re(p) > 0$.

Case (iv) For $b = d, k = 1, \alpha = \beta = 0$ *and* $\gamma = \delta$, (8) reduces to the generalized gamma function studied by Al-Musallam and Kalla [4, 5].

$$D\left(\begin{matrix}\lambda, a; c; \\ u, v\end{matrix} \quad p\right) := v^{-\lambda} \int_0^\infty t^{u-1} e^{-pt} 2F1(\lambda, a; c; -\frac{t}{v}) dt, \tag{12}$$

where $Re(u) > 0, Re(p) > 0, |argv| < \pi$.

Case (v) If we set $a = c, b = d$, p $= k = 1, \alpha = \beta = 0, \gamma = \delta$ an d $\lambda = m \in N_0$, equation (8) reduces to the generalized gamma function studied by Kobayashi [2, 3].

$$\Gamma_m(u, v) := \int_0^\infty \frac{t^{u-1} e^{-pt}}{(t+v)^m} dt, \tag{13}$$

For $m = 0$, (13) becomes the well-known gamma function.

$$\Gamma(m) := \int_0^\infty t^{u-1} e^{-pt} dt, \tag{14}$$

where $Re(u) > 0$.

Theorem 1. S^* is analytic in the domain $\Omega_u \times \Omega_v$.

The proof is similar to the corresponding theorem for the generalized gamma function given by Saxena and Kalla ([28], pp. 191-192), if we employ the Asymptotics estimate [Al-Musallam and Kalla (4)].

$${}_3R_2\left(\lambda, a, b; c, d; k; z\right) = A_1 z^{-\lambda} + A_2 z^{-\frac{a}{k}} + A_3 z^{-\frac{b}{k}} + O(z^{-\lambda-1}) + O(z^{-\frac{a}{k}-1}) + O(z^{-\frac{b}{k}-1}), \tag{15}$$

which holds for large Z, $|\arg(-z)| < \pi$. Here A_1, A_2, A_3 are numerical constants.

Lemma 1. The partial derivatives of S^* are:

$$\frac{\partial^n}{\partial u^n} S^* = v^{-\lambda} \cdot \int_0^\infty t^{u-1} M_{\alpha,\gamma}^{\beta,\delta}(-pt)(\log t)^n 3R2\left(\lambda, a, b; c, d; k; \frac{-t}{v}\right) dt, \tag{16}$$

and

$$\frac{\partial^n}{\partial v^n} S^* = (-1)^n (\lambda)_n S^*\left(\begin{matrix}\lambda + n, a, b; c, d; \alpha, \beta; \gamma, \delta; \\ u, v\end{matrix} \quad p, k\right). \tag{17}$$

The proof of (16) and (17) is trivial.

Lemma 2. Let $\lambda, \alpha, \beta, \gamma, \delta, a, b, c, d, p \in \mathbb{C}$ with $c, d \neq 0, -1, -2, \cdots$; $k > 0$ and Re(p) > 0, then following relation is valid:

$$S^*\left(\begin{matrix}\lambda, a, b; c, d; \alpha, \beta; \gamma, \delta; \\ u, v\end{matrix} p, k\right) = \frac{p\delta}{u\gamma} S^*\left(\begin{matrix}\lambda, a, b; c, d; \alpha, \beta; \gamma + 1, \delta + 1; \\ u + 1, v\end{matrix} p, k\right)$$

$$+ \frac{\lambda \Gamma(c)\Gamma(d)\Gamma(a+k)\Gamma(b+k)}{u\Gamma(a)\Gamma(b)\Gamma(c+k)\Gamma(d+k)} S^*\left(\begin{matrix}\lambda + 1, a + k, b + k; c + k, d + k; \alpha, \beta; \gamma, \delta; \\ u + 1, v\end{matrix} p, k\right). \tag{18}$$

Proof. If we use [7], equation (3.23)] for $\frac{d}{dz}[{}_3R_2^k(z)]$ and integrate by parts, then (8) reduces to (18).

4. The Generalized Incomplete Gamma Functions

For x, $k > 0$, we introduce the generalized incomplete gamma function in the form.

$$S_0^{*x}\left(\begin{matrix}\lambda, a, b; c, d; \alpha, \beta; \gamma, \delta; \\ u, v\end{matrix} \quad p, k\right) := v^{-\lambda} \int_0^x t^{u-1} M_{\alpha,\gamma}^{\beta,\delta}(-pt) 3R2(\lambda, a, b; c, d; k - \frac{t}{v}) dt, \tag{19}$$

where x, $k > 0$, Re(u) > 0, Re(p) > 0, and $|\arg v| < \pi$.

The generalized complementary incomplete gamma function is defined.

$$S_x^{*\infty}\left(\begin{matrix}\lambda, a, b; c, d; \alpha, \beta; \gamma, \delta; \\ u, v\end{matrix} \quad p, k\right) := v^{-\lambda} \int_x^\infty t^{u-1} M_{\alpha,\gamma}^{\beta,\delta}(-pt) 3R2(\lambda, a, b; c, d; k - \frac{t}{v}) dt, \tag{20}$$

where x, $k > 0$, Re(u) > 0, Re(p) > 0, $|\arg v| < \pi$.

Thus, the definitions (19) and (20) yield.

$$S^*\left(\begin{matrix}\lambda, a, b; c, d; \alpha, \beta; \gamma, \delta; \\ u, v\end{matrix} p, k\right) = S_0^{*x}\left(\begin{matrix}\lambda, a, b; c, d; \alpha, \beta; \gamma, \delta; \\ u, v\end{matrix} p, k\right) + S_x^{*\infty}\left(\begin{matrix}\lambda, a, b; c, d; \alpha, \beta; \gamma, \delta; \\ u, v\end{matrix} p, k\right), \tag{21}$$

Special cases:

Case (i) For $\alpha = \beta = 1$ and b=d equations (19) and (20) reduce to the results given by Saxena, Chena Ram, Naresh and Kalla [29].

Case (ii) For $\alpha = \beta = 0, \gamma = \delta$ *and* $b = d$,(19) and (20) reduce to the generalized incomplete gamma functions developed by Virchenko et al. ([7], p. 98).

Case (iii) Further for $b = d$, $\alpha = \beta = 0, \gamma = \delta$ and $k = 1$, (19) and (20) reduce to the incomplete gamma functions introduced by Al-Musallam and Kalla [4].

Case (iv) For $\alpha = \beta = 0$ *and* $\gamma = \delta$, (19) and (20) reduce to the results given by Chena Ram and Naresh [30].

Remark. If we set $a = c$, $b = d$, $\alpha = \beta = 0, \gamma = \delta$ and $p = k = 1$ in (19) and (20) and $\lambda \to 0$, then we find that

$$\operatorname*{Lt}_{\lambda\to 0} S^{*x}_{0}\begin{pmatrix}\lambda, a, b; a, b; 0,0; \gamma, \gamma; & 1,1\\ u, v & \end{pmatrix} = \gamma(u, x) = \int_0^x t^{u-1} e^{-t} dt, \tag{22}$$

where $\gamma(u, x)$ is the incomplete gamma function of the first kind, and

$$\operatorname*{Lt}_{\lambda\to 0} S^{*\infty}_{x}\begin{pmatrix}\lambda, a, b; a, b; 0,0, \gamma, \gamma; & 1,1\\ u, v & \end{pmatrix} = \Gamma(u, x) = \int_x^\infty t^{u-1} e^{-t} dt, \tag{23}$$

where $\Gamma(u, x)$ is the complementary incomplete gamma function of the second kind.

5. Probability Density Functions

Let m and ξ represents the shape parameters. Further, let σ and ρ denote the scale parameters. Then by taking $t = \sigma x^{\xi}$ and $dt = \sigma\, \xi\, x^{\xi-1} dx$, with $p = \frac{\rho}{\sigma}$ $(\rho > 0; \sigma > 0)$,

$$u = \frac{m+\xi}{\xi}\ (m + \xi > 0), \text{ and } v = \mathrm{n}\ (\mathrm{n} > 0)$$

Then (8) transform into the form.

$$\xi\sigma^{\frac{m+\xi}{\xi}} \int_0^\infty x^{m+\xi-1} M_{\alpha,\gamma}^{\beta,\delta}(-\rho x^{\xi}) 3\mathrm{R}2\left(\lambda, a, b; c, d; k - \frac{\sigma x^{\xi}}{n}\right) dx =$$

$$n^{\lambda} S^{*}\begin{pmatrix}\lambda, a, b; c, d; \alpha, \beta; \gamma, \delta; & \\ 1 + \frac{m}{\xi}, n & \frac{\rho}{\sigma}, k\end{pmatrix} (min\{\rho, \sigma, m + \xi\} > 0) \tag{24}$$

By virtue of integral formula (24), a class of probability density functions associated with the S^* -function can be defined by

$$f(x) := \begin{cases} \dfrac{\xi\, \sigma^{\frac{m+\xi}{\xi}}\, x^{m+\xi-1} M_{\alpha,\gamma}^{\beta,\delta}(-\rho x^{\xi})\ {}_3R_2\left(\lambda,a,b;c,d;k-\frac{\sigma x^{\xi}}{n}\right)}{n^{\lambda} S^{*}\begin{pmatrix}\lambda,a,b;c,d;\alpha,\beta;\gamma,\delta; & \\ 1+\frac{m}{\xi},n & \frac{\rho}{\sigma},k\end{pmatrix}} & (x > 0), \\ 0, elsewhere & \end{cases} \tag{25}$$

provided that the various parameters and variable x occurring in equation (25) are so constrained that the density function is always non-negative. It is evident that

$$\int_{-\infty}^{\infty} f(x) dx = 1. \tag{26}$$

We note that the behaviour of f(x) at zero depends on $m + \xi$.

$$f(0) = \xi\sigma^{\frac{1}{\xi}} n^{-\lambda}\left\{S^{*}\begin{pmatrix}\lambda, a, b; c, d; \alpha, \beta; \gamma, \delta; & \\ \frac{m}{\xi}, n & \frac{\rho}{\sigma}, k\end{pmatrix}\right\}^{-1} (m + \xi) = 1 \tag{27}$$

$$f(0) = 0\ (m + \xi) > 0$$

$$f(x) \to \infty \text{ as } x \to 0+ \text{ when } \mathrm{m} + \xi < 1, \tag{28}$$

$$\lim_{x\to\infty} f(x) = 0\ (\xi > 0), \tag{29}$$

It can be seen that

$$f'(x) = \left(\frac{m+\xi\text{-}1}{x} - \rho\xi\, x^{\xi-1} - \frac{\sigma\,\xi}{n}\, x^{\xi-1}\, \Psi\right) f(x), \tag{30}$$

Where, for convenience,

$$\Psi := \frac{\lambda\Gamma(c)\Gamma(d)\Gamma(a+k)\Gamma(b+k)}{\Gamma(a)\Gamma(b)\Gamma(c+k)\,\Gamma(d+k)} \frac{3\mathrm{R}2\left(\lambda+1,a+k,b+k;c+k,d+k;k;-\frac{\sigma x^{\xi}}{n}\right)}{3\mathrm{R}2\left(\lambda,a,b;c,d;k;-\frac{\sigma x^{\xi}}{n}\right)} \tag{31}$$

the formula (30) can be derived, if we differentiate both the sides of equation (25) with respect to x logarithmically and apply the following formula.

$$\frac{d}{dx}\left\{3\mathrm{R}2\left(\lambda,a;b;c,d;k-\frac{\sigma x^{\xi}}{n}\right)\right\} = -\frac{\sigma\xi\lambda}{n}\frac{\Gamma(c)\Gamma(d)\Gamma(a+k)\Gamma(b+k)}{\Gamma(a)\Gamma(b)\Gamma(c+k)\Gamma(d+k)}\, x^{\xi-1}$$

$$\times 3\mathrm{R}2\left(\lambda+1,a+k,b+k;c+k,d+k;k;-\frac{\sigma x^{\xi}}{n}\right) \tag{32}$$

Special Results:

For $\alpha=\beta=1$ and b=d, the results of this section reduce to the results given by Saxena, Chena Ram, Naresh and Kalla [29].

For $\alpha=\beta=0$ *and* $\gamma=\delta$, the results of the above section reduce to the results given by Chena Ram and Naresh [30].

For $\alpha=\beta=0, \gamma=\delta$ and b=d, the results of this section reduce to the generalized gamma function discussed by Virchenko et al. [6].

6. Some Statistical Functions

In this section, several basic statistical functions associated with the probability density function $f(x)$ defined by equation (25), will be evaluated.

6.1. The r^{th} Moment

The r^{th} moment μ_r^1 about the origin of a continuous real random variable X with the probability density function f(x) is given by

$$\mu_r^1 := \int_{-\infty}^{\infty} x^r f(x)\,dx =: \mathrm{E}[\mathrm{X}^r] \ (r \in N), \tag{33}$$

which on using equation (24) and definition (25) gives

$$\mu_r^1 = \sigma^{-\frac{r}{\xi}} S^*\begin{pmatrix}\lambda,a,b;c,d;\alpha,\beta;\gamma,\delta; & \\ 1+\frac{m+r}{\xi},n & \frac{\rho}{\sigma},k\end{pmatrix}\left\{S^*\begin{pmatrix}\lambda,a,b;c,d;\alpha,\beta;\gamma,\delta; & \\ 1+\frac{m}{\xi},n & \frac{\rho}{\sigma},k\end{pmatrix}\right\}^{-1} \tag{34}$$

In particular, for r=1, the expected value of the random variable X (also referred to as the mean or the first moment of X) is obtained as

$$E(x) := \int_{-\infty}^{\infty} x\,f(x)\,dx = \sigma^{-\frac{1}{\xi}} S^*\begin{pmatrix}\lambda,a,b;c,d;\alpha,\beta;\gamma,\delta; & \\ 1+\frac{m+1}{\xi},n & \frac{\rho}{\sigma},k\end{pmatrix}\left\{S^*\begin{pmatrix}\lambda,a,b;c,d;\alpha,\beta;\gamma,\delta; & \\ 1+\frac{m}{\xi},n & \frac{\rho}{\sigma},k\end{pmatrix}\right\}^{-1} \tag{35}$$

6.2. The Moment Generating Function

The moment generating function $M(t;\xi)$ of a continuous random variable X having the probability density function $f(x)$ is defined by

$$M(t;\xi) = E[e^{tX}] := \int_{-\infty}^{\infty} e^{tx} f(x)dx = \frac{\xi\sigma^{\frac{m}{\xi}+1}\int_0^{\infty} t^{u-1} M_{\alpha,\gamma}^{\beta,\delta}(-pt)\,3\mathrm{R}2(\lambda,a,b;c,d;k-\frac{t}{v})dt,}{n^{\lambda} S^*\begin{pmatrix}\lambda,a,b;c,d;\alpha,\beta;\gamma,\delta; & \\ 1+\frac{m}{\xi},n & \frac{\rho}{\sigma},k\end{pmatrix}} \tag{36}$$

which itself is a generalization of a result given by Saxena, Chena Ram, Naresh and Kalla [29].

For $\alpha=\beta=0$ *and* $\gamma=\delta$, the results of the above section reduce to the results given by Chena Ram and Naresh [30].

If we set $\alpha=\beta=0, \gamma=\delta$ *and* $b=d$ and k=1 (27) reduces to the moment generating function studied by Kalla et al. [31].

6.3. The Hazard Rate Function

For a continuous random variable X having the probability density function f(x), the cumulative distribution function F(t) is given by

$$F(t) := \int_{-\infty}^{t} f(x)\,dx =: Pr\,ob\{X \in (-\infty,t)\}, \tag{37}$$

that is, by

$$F(t) = {S^*}_0^{\sigma t^{\xi}}\begin{pmatrix}\lambda, a, b; c, d; \alpha, \beta; \gamma, \delta; & \\ \frac{m}{\xi}+1, n & \frac{\rho}{\sigma}, k\end{pmatrix}\left\{S^*\begin{pmatrix}\lambda, a, b; c, d; \alpha, \beta; \gamma, \delta & \\ \frac{m}{\xi}+1, n & ; \quad \frac{\rho}{\sigma}, k\end{pmatrix}\right\}^{-1}, \tag{38}$$

where we have used the definition (19) and (25). Thus by virtue of the relationship (21), the hazard (or failure) rate function $h(t)$ is given by

$$h(t) \coloneqq \frac{f(t)}{1-F(t)} = \xi\sigma^{\frac{m+\xi}{\xi}} n^{-\lambda} t^{m+\xi-1} M_{\alpha,\gamma}^{\beta,\delta}(-\rho x^{\xi})\ {}_3R_2\left(\lambda, a, b; c, d; k - \frac{\sigma x^{\xi}}{n}\right) \times$$

$$\left\{{S^*}_{\sigma t^{\xi}}^{\infty}\begin{pmatrix}\lambda, a, b; c, d; \alpha, \beta; \gamma, \delta; & \\ , 1+\frac{m}{\xi} v & p, k\end{pmatrix}\right\}^{-1} \quad (t > 0), \tag{39}$$

in terms of the complementary incomplete S^*-function defined by equation (20).

In passing, we remark that the above derivations would also give the survival (or reliability) functions.

$$S(t) \coloneqq 1 - F(t) = \int_t^{\infty} f(x)\, dx = {S^*}_{\sigma t^{\xi}}^{\infty}\begin{pmatrix}\lambda, a, b; c, d; \alpha, \beta; \gamma, \delta; & \\ 1+\frac{m}{\xi}, n & \frac{\rho}{\sigma}, k\end{pmatrix}\left\{S^*\begin{pmatrix}\lambda, a, b; c, d; \alpha, \beta; \gamma, \delta & \\ \frac{m}{\xi}+1, n & ; \quad \frac{\rho}{\sigma}, k\end{pmatrix}\right\}^{-1} \quad (t > 0) \tag{40}$$

6.4. The Mean Residual Life (or Remaining Life Expectancy) Function

For a continuous random variable X, the mean residual life (or remaining life expectancy) function K(t) is given by

$$K(t) := E[X - t | X \geq t] = \frac{1}{S(t)} \int_t^{\infty} (x - t) f(x)\ dx, \tag{41}$$

$$= \frac{1}{S(t)} \int_t^{\infty} x\, f(x)\ dx - t, \tag{42}$$

since $S(t)$ denotes the survivor (or reliability) function denoted by equation (40).

By virtue of the definition (25), if we use the substitution $z = \sigma\, x^{\xi}$ *and*
$dz = \sigma\, \xi\, x^{\xi-1} dx$, the equation (42)

$$\int_t^{\infty} x\ f(x)\ dx = \sigma^{-\frac{1}{\xi}} {S^*}_{\sigma t^{\xi}}^{\infty}\begin{pmatrix}\lambda, a, b; c, d; \alpha, \beta; \gamma, \delta; & \\ \left(\frac{1+m}{\xi}\right), n & \frac{\rho}{\sigma}, k\end{pmatrix}\left\{S^*\begin{pmatrix}\lambda, a, b; c, d; \alpha, \beta; \gamma, \delta & \\ \frac{m}{\xi}+1, n & ; \frac{\rho}{\sigma}, k\end{pmatrix}\right\}^{-1} \tag{43}$$

so that

$$K(t) = \sigma^{-\frac{1}{\xi}} {S^*}_{\sigma t^{\xi}}^{\infty}\begin{pmatrix}\lambda, a, b; c, d; \alpha, \beta; \gamma, \delta; & \\ \left(\frac{1+m}{\xi}\right), n & \frac{\rho}{\sigma}, k\end{pmatrix}\left\{S^*\begin{pmatrix}\lambda, a, b; c, d; \alpha, \beta & \\ \frac{m}{\xi}+1, n & ; \frac{\rho}{\sigma}, k\end{pmatrix}\right\}^{-1} - t. \tag{44}$$

In terms of the complementary incomplete S^*-function defined by the equation (20).

7. Conclusions

The results presented this paper by using confluent hypergeometric Mittage-Leffler functions are new and potential useful. Density function derived (25), which includes various density functions are interesting application for the evaluation of problem arising in probability models.

References

[1] Saxena, R. K, Chena Ram and Naresh, Some results associated with a generalized hypergeometric function, Bulletin of pure and applied sciences. Vol. 24e (No. 2) 2005: P. 305-316.

[2] Kobayashi, K., On generalized Gamma functions occurring in diffraction theory. *Journal of the Physical Society of Japan*, 60 (1991), 1501–1512.

[3] Kobayashi, K., Plane wave diffraction by a strip: Exact and asymptotic solutions. *Journal of the Physical Society of Japan*, 60 (1991), 1891–1905.

[4] Al-Musallam, F., Kalla, S. L., Asymptotic expansions for generalized gamma and incomplete gamma function, Appl. Anal., 66 (1997), 173-187.

[5] Al-Musallam, F., Kalla, S. L., Further results on a generalized gamma function occurring in diffraction theory. Integral Transforms and Special function, 7 (1998), 175-190.

[6] Virchenko, N., Kalla, S. L., On some generalization of the functions of hypergeometric type, Fract. Calc and Appl. Anal. 2 (3) (1999), 233-244.

[7] Virchenko, N., Kalla, S. L. and Al-Zamel, A., Some results on a generalized hypergeometric function. *Integral Transforms and Special Functions*, 12, 89–100 (2001).

[8] Ghanim, F.; Al- Janaby, H. F. Inclusion and Convolution Features of Univalent Meromorphic functions Correlating with Mittage-Leffler function. Filomat 2020, 34, 2141-2150.

[9] Ghanim, F.; Al- Janaby, H. F. Some analytical merites of Kummer-Type function associated with Mittage-Leffler parameters. Arab. J. Basic Appl. Sci. 2021, 28, 255-263.

[10] Ghanim, F.; Bendak, S.; Hawarneh, A. A. Fractional Calculus Operator involving Mittage-Leffler confluent hypergeometric function. 2021, preprint.

[11] Oros, G. I. Applications of inequalities in the complex plane associated with confluent hypergeometric function. Symmetry 2021, 13, 259.

[12] Oros, G. I. Study on new integral operators defined using confluent hypergeometric function. Adv. Differ. Equ. 2021, 1, 1-11.

[13] Haubold, H. J.; Mathai, A. M.; Saxena, R. K. Mittage-Leffler functions and their applications. J. Appl. Math. 2011, 298628.

[14] Paneva-Konovska, J. Differential and integral relations in the class of muli-index Mittage-Leffler functions. Fract. Calc. Appl. Anal. 2018, 21, 254-265.

[15] Mainardi, F.; Gorenflo, R. On Mittage-Leffler -type functions in fractional evolution processes. J. Comput. A ppl. Math. 2000, 118, 283-299.

[16] Mathai, A. M.; Haubold, H. J. Mittage-Leffler functions and fractional calculus. In Special functions for applied scientists; Springer: New York, USA, 2008; pp. 79-134.

[17] Srivastava, H. M. On an extension of the Mitage-Leffler function. Yokohama Math. J. 1968, 16, 77-88.

[18] Srivastava, H. M. Some families of Mittage-Leffler type functions and associated operators of fractional calculus (Survey). TWMS J. Pure Appl. Math. 2016, 7, 123-145.

[19] Bazighifan, O.; Dassios, I. Riccati technique and asymptotic behaviour of fourth order advanced differential equations. Mathematics 2020, 8, 590.

[20] Mittage-Leffler, G. M. Sur la nouvelle function Eα(x). C. R. Acad. Sci. Paris 1903, 137, 554-558.

[21] Mittage-Leffler, G. M. Sur la repr'estation analytique d; une branche uniforme d'une function monogene: Cinquieme note. Acta Math. 1905, 29, 101-181.

[22] Wiman, A. ÜBer den fundamentalsatz in der theorie der funcktionen Eα(x). Acta Math. 1905, 29, 191-201.

[23] Wiman, A. ÜBer die nullstellen der funktionen Eα(x). Acta Math. 1905, 29, 217-234.

[24] Gorenflo, R.; Kilbas, A. A.; Mainardi, F.; Rogosin, S. V. Mittage-Leffler functions, Related Topic and Applications; Springer: Berlin/ Heidelberg, Germany, 2016.

[25] Srivastava, H. M.; Tomovski, Ž. Fractional calculus with an integral operator containing a generalized Mittage-Leffler function in the kernel. J. Appl. Math. Comput. 2009, 211, 198-210.

[26] E. Ghanim, Hiba f. AL-Janaby, Omar Bazighifan, Some new extensions on fractional differential and integral properties for Mittage-Leffler confluent hypergeometric function. Fractal and Fractional, 2021, 5, 143.

[27] Bateman, H.; Erdélyi, A. Higher Transcendental functions; McGraw Hill: New York, NY, USA; Toronto, On, Canada; London, UK, 1955; Volume 3.

[28] Saxena, R. K and Kalla, S. L., On a generalized gamma function occurring in differaction theory, Int. J. Appl. Math, 5, 18, (2001).

[29] Saxena, R. K., Kalla S. L. Chena Ram and Naresh., Generalized gamma-type functions involving Kummers's confluent hypergeometric function and associated probability distributions. Integral Transforms and Special Functions. Vol 18, No. 9, September 2007, 679-687.

[30] Chena Ram, Naresh.; Certain properties of generalized gamma function. Acta Cienica Indica, Vol. XXXVII M, No. 1, 33 (2011).

[31] Kalla, S. L., Al-Saqabi, B. N., Khajah, H. G., A unifed form of gamma –type distributions, Appl. Math. comput., 118 (2001), 175-187.