

Research Article

Boundary Domain Integral Equations for Variable Coefficient Mixed BVP in 2D Unbounded Domain

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Abstract

In this paper, the direct segregated Boundary Domain Integral Equations (BDIEs) for the Mixed Boundary Value Problems (MBVPs) for a scalar second order elliptic Partial Differential Equation (PDE) with variable coefficient in unbounded (exterior) 2D domain is considered. Otar Chkadua, Sergey Mikhailov and David Natroshvili formulated both the interior and exterior 3D domain of the direct segregated systems of BDIEs for the MBVPs for a scalar second order elliptic PDE with a variable coefficients. On the other hand Sergey Mikhailov and Tamirat Temesgen formulated only the interior 2D domain of the direct segregated systems of BDIEs for the MBVPs for a scalar second order divergent elliptic PDE with a variable coefficients. However, in this paper we formulated the exterior 2D domain of the direct segregated systems of BDIEs for the MBVPs for a scalar second order divergent elliptic PDE with a variable coefficients. The aim of this work is to reduce the MBVPs to some direct segregated BDIEs with the use of an appropriate parametrix (Levi function). We examine the characteristics of corresponding parametrix-based integral volume and layer potentials in some weighted Sobolev spaces, as well as the unique solvability of BDIEs and their equivalence to the original MBVPs. This analysis is based on the corresponding properties of the MBVPs in weighted Sobolev spaces that are proved as well.

Keywords

Partial Differential Equation, Variable Coefficient, Unbounded Domain, Weighted Sobolev Spaces, Parametrix (Levi Function), Single Layer Potential, Mixed Boundary Value Problem, Boundary-Domain Integral Equations

1. Introduction

Mathematical modeling of inhomogeneous media (such as functionally graded materials or materials with damage-induced inhomogeneity) in solid mechanics, electromagnetics, thermo-conductivity, fluid flows through porous media, and other branches of physics and engineering frequently involves PDEs with variable coefficients. When the PDE coefficients are not constant, there are typically no explicit fundamental solutions available, which makes it impossible to solve BVPs for such PDEs numerically. However,

for an extensive range of variable-coefficient PDEs, an explicit parametrix (Levi function) linked to a fundamental solution of corresponding frozen coefficient PDEs can be utilized instead. This reduces BVPs for these PDEs in interior domains to BDIE systems for additional numerical solution of the latter. (e.g., [3, 5, 15, 16, 17, 20]).

The primary objective of this study is to demonstrate the reduction of mixed problems with variable coefficients in exterior domains to certain systems of BDIEs. Additionally,

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Received: 26 July 2024; **Accepted:** 20 August 2024; **Published:** 30 August 2024



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we aim to explore the distinct solvability of BDIEs and their equivalence to the original BVP in the weighted Sobolev spaces. To achieve this, we extend to exterior domains and weighted spaces, using the techniques developed in [20] for interior domains and standard Sobolev (Bessel potential) spaces.

The characteristics of the related boundary value problems are crucial to the BDIE analysis. Today, a lot of research has been done on variable-coefficient BVPs in bounded domains, (e.g., [7, 9]). In particular, the analysis of segregated boundary-domain integral equations for variable-coefficient MBVPs 3D unbounded domains can be found in [4]. Nonetheless, due to the logarithmic term in the parametrix of the related partial differential equation, the BDIEs in the 2D case show unique characteristics when compared to the higher dimensions. As a result, in order to guarantee the invertibility of the layer potentials and, consequently, the BDIEs unique solvability, we must impose requirements on the function spaces.

2. Preliminaries

Let $\Omega = \Omega^+$ be an unbounded open domain in \mathbb{R}^2 such that the complement $\Omega^- := \mathbb{R}^2 \setminus \bar{\Omega}$ is bounded open domain. Let the boundary $\partial\Omega = \partial\Omega^-$ be closed and infinitely smooth curve. The space of infinitely differentiable functions having compact support in Ω is denoted by $\mathcal{D}(\Omega)$ and its dual space, the space of distributions, by $\mathcal{D}'(\Omega)$, while $\mathcal{D}(\bar{\Omega})$ is the set of restrictions on $\bar{\Omega}$ of functions from $\mathcal{D}(\mathbb{R}^2)$. The spaces $H^s(\Omega), H^s(\partial\Omega)$ denote the Sobolev (Bessel potential) spaces. We also denote $\tilde{H}^s(\Gamma_1) = \{g: g \in H^s(\Gamma), \text{supp } g \subset \bar{\Gamma}_1\}$, $H^s(\Gamma_1) = \{r_{\Gamma_1}g: g \in H^s(\Gamma)\}$, where Γ_1 is a proper submanifold of a closed surface Γ and r_{Γ_1} is the restriction operator on Γ_1 . Moreover for $s = -\frac{1}{2}$ we define the subspace $H_{**}^{-\frac{1}{2}}(\Gamma_1)$ of $H^{-\frac{1}{2}}(\Gamma_1)$ as $H_{**}^{-\frac{1}{2}}(\Gamma_1) := \{r_{\Gamma_1}g: g \in H^{-\frac{1}{2}}(\Gamma): \langle g, 1 \rangle_{\Gamma} = 0\}$. We shall consider the following second order partial differential equation, with variable coefficient

$$Au(x) := \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left(a(x) \frac{\partial u(x)}{\partial x_i} \right) = f(x) \quad x \in \Omega \quad (1)$$

where u is unknown function; $f(x)$ and $a(x) > a_0 > 0$ are given functions in Ω . We will further use the weighted Sobolev spaces.

Let

$$\rho(x) := (1 + |x|^2)^{1/2} \ln(2 + |x|^2)$$

For any real α , we denote by $L_2(\rho^\alpha; \Omega)$ the weighted Lebesgue space (e.g., [8]) consisting of all measurable functions $g(x)$ on Ω such that $g\rho^\alpha \in L_2(\Omega)$, i.e.,

$$\|g\|_{L_2(\rho^\alpha; \Omega)} = \left[\int_{\Omega} |g(x)\rho^\alpha(x)|^2 dx \right]^{\frac{1}{2}} < \infty$$

The space $L_2(\rho^\alpha; \Omega)$, equipped with the norm $\|\cdot\|_{L_2(\rho^\alpha; \Omega)}$ and appropriate inner product, is a Hilbert space. The weighted Sobolev space $\mathcal{H}^1(\Omega)$ is defined by

$$\mathcal{H}^1(\Omega) := \{g \in L_2(\rho^{-1}; \Omega): \nabla g \in L_2(\Omega)\} \quad (2)$$

and for its norm we have $\|g\|_{\mathcal{H}^1(\Omega)}^2 = \|g\|_{L_2(\rho^{-1}; \Omega)}^2 + \|\nabla g\|_{L_2(\Omega)}^2$, while $|g|_{\mathcal{H}^1(\Omega)}^2 := \sum_{i=1}^2 \int_{\Omega} \left| \frac{\partial g}{\partial x_i} \right|^2 dx = \|\nabla g\|_{L_2(\Omega)}^2$ is the square of the semi-norm. The space $\mathcal{D}(\mathbb{R}^2)$ is dense in $\mathcal{H}^1(\mathbb{R}^2)$, (e.g., Theorem 7.2 in [1]). This implies that the dual space of $\mathcal{H}^1(\mathbb{R}^2)$, denoted by $\mathcal{H}^{-1}(\mathbb{R}^2)$, is a space of distributions. It is possible to show that the space $\mathcal{D}(\bar{\Omega})$ is dense in $\mathcal{H}^1(\Omega)$ by using the corresponding property of the space $H^1(\Omega)$. The trace operator γ^+ on $\partial\Omega$ defined on functions from $\mathcal{H}^1(\Omega)$, satisfies the usual trace theorems. This allows to define in particular the subspace

$$\mathcal{H}_0^1(\Omega) = \{g \in \mathcal{H}^1(\Omega): \gamma^+ g = 0\}$$

It can be proved that $\mathcal{D}(\Omega)$ is dense in $\mathcal{H}_0^1(\Omega)$ and therefore, its dual space is a space of distributions. Let us denote by $\tilde{\mathcal{H}}^1(\Omega)$ a completion of $\mathcal{D}(\Omega)$ in $\mathcal{H}^1(\mathbb{R}^2)$, and $\tilde{\mathcal{H}}^{-1}(\Omega) := [\mathcal{H}^1(\Omega)]'$, $\mathcal{H}^{-1}(\Omega) := [\tilde{\mathcal{H}}^1(\Omega)]'$ are the corresponding dual spaces. The inclusion $L_2(\rho; \Omega) \subset \mathcal{H}^{-1}(\Omega)$ holds and a distribution f in the dual space $\tilde{\mathcal{H}}^{-1}(\Omega)$ has the form $f = \sum_{i=1}^2 \frac{\partial g_i}{\partial x_i} + f_0$, where $g_i \in L_2(\mathbb{R}^2)$ and is zero outside Ω , $f_0 \in L_2(\rho; \Omega)$, (e.g., Eq. (2.5.129) in [13]). This implies that $\mathcal{D}(\Omega)$ is dense in $\tilde{\mathcal{H}}^{-1}(\Omega)$ and $\mathcal{D}(\mathbb{R}^2)$ is dense in $\mathcal{H}^{-1}(\mathbb{R}^2)$.

Lemma 1. The space $\mathcal{H}^1(\Omega)$ contains constant functions.

Proof. Let $C \in \mathbb{R}$ then from Definition 2.2 the result follows.

Note That 1. Lemma 1 implies that, the space of real constants, \mathbb{R} , is a closed subspace of $\mathcal{H}^1(\Omega)$. Hence we can define the quotient space $\mathcal{H}^1(\Omega)/\mathbb{R}$, which is a complete normed space, and its norm is given by $\|u + \mathbb{R}\|_{\mathcal{H}^1(\Omega)/\mathbb{R}} = \inf_{c \in \mathbb{R}} \|u + c\|_{\mathcal{H}^1(\Omega)}$. The dual space $(\mathcal{H}^1(\Omega)/\mathbb{R})'$ is identified with $\tilde{\mathcal{H}}^{-1}(\Omega) \perp \mathbb{R}$, i.e., $(\mathcal{H}^1(\Omega)/\mathbb{R})' = \tilde{\mathcal{H}}^{-1}(\Omega) \perp \mathbb{R}$ since they are isometrically isomorphic (e.g., Lemma 2.12(ii) in [9]). Similarly, $(\tilde{\mathcal{H}}^1(\Omega)/\mathbb{R})' = \mathcal{H}^{-1}(\Omega) \perp \mathbb{R}$. The following Poincaré-type inequalities hold (e.g., Theorems 1.1 and 1.2 in [2]).

Theorem 1. (i) The semi-norm $|\cdot|_{\mathcal{H}^1(\Omega)}$ defined on $\mathcal{H}^1(\Omega)/\mathbb{R}$ is a norm equivalent to the quotient norm, i.e., there exist positive constants k_1, K_1 such that

$$k_1 |v|_{\mathcal{H}^1(\Omega)} \leq \|v\|_{\mathcal{H}^1(\Omega)/\mathbb{R}} \leq K_1 |v|_{\mathcal{H}^1(\Omega)}$$

(ii) Moreover, the semi-norm $|\cdot|_{\mathcal{H}^1(\Omega)}$ is a norm on

$\mathcal{H}_0^1(\Omega)$ equivalent to the norm $\|\cdot\|_{\mathcal{H}^1(\Omega)}$, i.e., there exist positive constants k_2, K_2 such that

$$k_2|v|_{\mathcal{H}^1(\Omega)} \leq \|v\|_{\mathcal{H}_0^1(\Omega)} \leq K_2|v|_{\mathcal{H}^1(\Omega)}$$

For $u \in \mathcal{H}^1(\Omega)$ and the coefficient $a(x) \in L_\infty(\Omega)$, PDE (1) is well defined in the sense of distribution as $\langle Au, v \rangle_\Omega := -\langle a \nabla u, \nabla v \rangle_\Omega = -\mathcal{E}(u, v)$, for any $v \in \mathcal{D}(\Omega)$, where $\mathcal{E}(u, v) := \int_\Omega E(u, v)(x) dx$, $E(u, v)(x) := \nabla v(x) \cdot a(x) \nabla u(x)$. From here on, unless specified otherwise, we presume that there exist some constants a_0, a_1 such that

$$a \in L_\infty(\mathbb{R}^2) \text{ and } 0 < a_0 < a(x) < a_1 < \infty \text{ for } x \in \mathbb{R}^2 \quad (3)$$

To obtain boundary-domain integral equations, we will also always consider the coefficient a such that

$$a \in C^1(\mathbb{R}^2) \text{ and } \rho \nabla a \in L_\infty(\mathbb{R}^2) \quad (4)$$

(e.g., [7]) for $u \in H^1(\Omega)$, if $u \in \mathcal{H}^1(\Omega^+)$, then from the trace theorem it follows that, $\gamma^+ u \in H^{\frac{1}{2}}(\partial\Omega)$, where $\gamma^+ = \gamma_{\partial\Omega}^+$ is the trace operator on $\partial\Omega$ from the exterior domain Ω^+ .

For the operator A , similar to [4] for the three dimensional case, we introduce the space, $\mathcal{H}^{1,0}(\Omega; A) := \{g \in \mathcal{H}^1(\Omega) : Ag \in L_2(\rho; \Omega)\}$, where the norm is given by its square, $\|g\|_{\mathcal{H}^{1,0}(\Omega; A)}^2 := \|g\|_{\mathcal{H}^1(\Omega)}^2 + \|Ag\|_{L_2(\rho; \Omega)}^2$. For $u \in \mathcal{H}^{1,0}(\Omega; A)$, as in the 3 D case, [4], we define the canonical co-normal derivative $T^+ u \in H^{-\frac{1}{2}}(\partial\Omega)$ similar to, for example in Lemma 3.2 of [6] and Lemma 4.3 of [9] as

$$\langle T^+ u, \omega \rangle_{\partial\Omega} := \int_\Omega [(\gamma_{-1}^+ \omega) Au + E(u, \gamma_{-1}^+ \omega)] dx \quad \forall \omega \in H^{\frac{1}{2}}(\partial\Omega)$$

where $\gamma_{-1}^+ : H^{\frac{1}{2}}(\partial\Omega) \rightarrow \mathcal{H}^1(\Omega)$ is a bounded right inverse to the trace operator $\gamma^+ : \mathcal{H}^1(\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$, and $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the duality brackets between the spaces $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$ which extends the usual $L_2(\partial\Omega)$ scalar product. The operator $T^+ : \mathcal{H}^{1,0}(\Omega; A) \rightarrow H^{-\frac{1}{2}}(\partial\Omega)$ is continuous and gives the continuous extension to $\mathcal{H}^{1,0}(\Omega; A)$ of the classical co-normal derivative operator $a \frac{\partial}{\partial n}$, where $\frac{\partial}{\partial n} = \gamma^+ \nabla \cdot n$ and $n = n^+$ is normal vector on $\partial\Omega$ directed outward the exterior domain Ω . When $a \equiv 1$, we employ for T^+ the notation T_Δ^+ , which is the continuous extension on $\mathcal{H}^{1,0}(\Omega; \Delta)$ of the classical normal derivative operator ∂_n . Similar to the proofs available in Lemma 3.4 of [6] (for the spaces $H^{s,t}(\Omega; A)$ see also [11]), one can show that for $u \in \mathcal{H}^{1,0}(\Omega; A)$ and $v \in \mathcal{H}^1(\Omega)$ the first Green identity

$$\langle T^+ u, \gamma^+ v \rangle_{\partial\Omega} = \int_\Omega [v Au + E(u, v)] dx \quad \forall v \in \mathcal{H}^1(\Omega) \quad (5)$$

holds true. Then, for any chooses of $u, v \in \mathcal{H}^{1,0}(\Omega; A)$ we obtain the second Green identity,

$$\int_\Omega [v Au - u Av] dx = \langle T^+ u, \gamma^+ v \rangle_{\partial\Omega} - \langle T^+ v, \gamma^+ u \rangle_{\partial\Omega} \quad (6)$$

Remark 1. If a satisfies condition (3) and the second condition in (4), then $\|ga\|_{\mathcal{H}^1(\Omega)} \leq C_1 g\|_{\mathcal{H}^1(\Omega)}, \|g\|_{\mathcal{H}^1(\Omega)} \leq C_2 \|g\|_{\mathcal{H}^1(\Omega)}$, where the constant C_1 and C_2 are independent of $g \in \mathcal{H}^1(\Omega)$, which means, a and $1/a$ are multipliers in the space $\mathcal{H}^1(\Omega)$.

3. Mixed BVP in Exterior Domains

Let $\partial\Omega = \overline{\partial\Omega}_D \cup \overline{\partial\Omega}_N$, where $\partial\Omega_D$ and $\partial\Omega_N$ are relatively open, non-empty and non-intersecting parts of $\partial\Omega$. We will derive and analyze the system of BDIEs for the following mixed BVP: Given $f \in L_2(\rho; \Omega) \perp \mathfrak{R}, \psi_0 \in H_{**}^{-\frac{1}{2}}(\partial\Omega_N)$ and $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$, find a function $u \in \mathcal{H}^{1,0}(\Omega; A)$ such that:

$$Au = f \text{ in } \Omega \quad (7)$$

$$\gamma^+ u = \varphi_0 \text{ on } \partial\Omega_D \quad (8)$$

$$T^+ u = \psi_0 \text{ on } \partial\Omega_N \quad (9)$$

Here $\partial\Omega = \overline{\partial\Omega}_N \cup \overline{\partial\Omega}_D$, while $\partial\Omega_D \neq \emptyset$ and $\partial\Omega_N \neq \emptyset$ are nonintersecting simply connected sub-manifolds of $\partial\Omega$ with an infinitely smooth boundary curve $\ell = \overline{\partial\Omega}_N \cap \overline{\partial\Omega}_D$.

Let us denote by $\mathcal{A}_M = [A, T^+, \gamma^+]^T : \mathcal{H}^{1,0}(\Omega; A) \rightarrow L_2(\rho; \Omega) \perp \mathfrak{R} \times H_{**}^{-\frac{1}{2}}(\partial\Omega_N) \times H^{\frac{1}{2}}(\partial\Omega_D)$, the operator on the left, which is obviously continuous. As with the three-dimensional case's proof in [4], one can show the following assertion in the 2D case. (e.g; Theorem 8.6 in [4]).

Theorem 2. Under conditions (3), the Mixed problem (7)-(9) is uniquely solvable and its solution can be written as $u = \mathcal{A}_M^{-1}(f, \psi_0, \varphi_0)^T$, where the operator $\mathcal{A}_M^{-1} : L_2(\rho; \Omega) \times H^{-\frac{1}{2}}(\partial\Omega_N) \times H^{\frac{1}{2}}(\partial\Omega_D) \rightarrow \mathcal{H}^{1,0}(\Omega; A)$ is continuous.

4. Parametrix-Based Potentials in Exterior Domain

A function $P(x, y)$ is a parametrix (Levi function) for the operator A if $A_x P(x, y) = \delta(x - y) + R(x, y)$, where δ is the Dirac-delta distribution, while $R(x, y)$ is a remainder possessing at most a weak (integrable) singularity at $x = y$. In particular, (e.g., [10]) the function

$$P(x, y) = \frac{\ln|x-y|}{2\pi a(y)}, \quad x, y \in \mathbb{R}^2 \quad (10)$$

is a parametrix for the operator $A(x, \partial_x)$ given by:

$$A(x, \partial_x)P(x, y) = R(x, y) + \delta(x - y) \quad (11)$$

$$\text{Where } R(x, y) = \sum_{i=1}^2 \frac{x_i - y_i}{2\pi a(y)|x - y|^2} \frac{\partial a(x)}{\partial x_i}, \quad x, y \in \mathbb{R}^2 \quad (12)$$

Let $u \in \mathcal{D}(\bar{\Omega})$. For any fixed $y \in \Omega$, let $B_\varepsilon(y)$ be an open ball centered at y with a sufficiently small radius $\varepsilon > 0$, and let $B_r(0)$ be an open ball centered at the origin with a radius r large enough to contain $\partial\Omega$ and the support of u , put $\Omega_\varepsilon := (\Omega \cap B_r(0)) \setminus B_\varepsilon(y)$, we have $R(\cdot, y) \in L_2(\rho; \Omega_\varepsilon)$ and thus $P(\cdot, y) \in \mathcal{H}^{1,0}(\Omega_\varepsilon)$ by (11). Applying the second Green identity (6) in Ω_ε with $v = P(y, \cdot)$ and taking usual limits as $\varepsilon \rightarrow 0$, (eg., [12]), we get the third Green identity in $\Omega_r := \Omega \cap B_r(0)$,

$$u + Ru - V(T^+u) + W(\gamma^+u) = PAu \quad (13)$$

for $u \in \mathcal{D}(\bar{\Omega})$.

Here,

$$\mathcal{R}g(y) := \int_{\Omega} R(x, y)g(x)dx, \quad (14)$$

$$\mathcal{P}g(y) := \int_{\Omega} P(x, y)g(x)dx, \quad y \in \mathbb{R}^2 \quad (15)$$

are, respectively, the remainder potentials and parametrix-based Newtonian, while

$$Vg(y) := -\int_{\partial\Omega} [P(x, y)]g(x)dS_x,$$

$$Wg(y) := -\int_{\partial\Omega} [T_x P(x, y)]g(x)dS_x, \quad y \in \mathbb{R}^2 \setminus \partial\Omega$$

are the parametrix-based single layer and double layer potentials. Deducing (13) we took into account that $u \equiv 0$ in $\Omega \setminus B_r(0) \subset \Omega \setminus \text{supp } u$. Since no term in (13) depends on r if r is sufficiently large, we obtain that (13) is valid in the whole domain Ω for any $u \in \mathcal{D}(\bar{\Omega})$.

From definitions (10)-(12) and (14)-(15) The parametrix-based potential operators can be represented in terms of their corresponding ones for $a = 1$ (i.e., associated with the Laplace operator Δ), (eg., [3, 4]),

$$\mathcal{P}g = \frac{1}{a}\mathcal{P}_\Delta g, \quad Vg = \frac{1}{a}V_\Delta g,$$

$$Wg = \frac{1}{a}W_\Delta(ag), \quad \mathcal{R}g = -\frac{1}{a}\sum_{j=1}^2 \partial_j [\mathcal{P}_\Delta(g \partial_j a)] \quad (16)$$

The Newtonian and the remainder potential operators given by (14) for $\Omega = \mathbb{R}^2$ will be denoted as P and R , respectively, and the relations similar to (16) hold for them as well.

5. Invertibility of the Single Layer Potential Operator

The boundary integral operator $\mathcal{V}_\Delta: H^{-1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$ is Fredholm operator of index zero (e.g., Theorem

7.6 in [9]). Thus the relation (16), leads to the same result for single layer potential \mathcal{V} . For the 3-D case, the following holds. For $\psi^* \in H^{-1/2}(\partial\Omega)$, if $V\psi^*(y) = 0, y \in \Omega$, then $\psi^* = 0$, which implies the invertibility of single layer potential operator mapping from $H^{-1/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$. But it is not true in the two dimensional case. It is well known for some 2D domains the kernel of the operator \mathcal{V}_Δ is non-zero, which by (16) also implies that $\ker \mathcal{V} \neq \{0\}$ for the same domains. The following example illustrates this fact.

Example 1. Take the density function $\phi \equiv 1$ and $\Omega = B(0, R)$ to be a disc of radius R centered at the origin and $\partial\Omega = \partial B(0, R)$ be the circular boundary of the disc. We can show that

$$a(y)V\phi(y) = V_\Delta\phi(y) = \begin{cases} R\ln|y|, & \text{for } |y| > R, \\ R\ln R, & \text{for } |y| \leq R \end{cases}$$

Proof. Let $\phi \equiv 1$. Then

$$V_\Delta\phi(y) = \frac{1}{2\pi} \int_{|x|=R} \ln|y - x|dS_x$$

If $|y| > R$, then the function $g(x) = \ln|y - x|$ is harmonic in the disk $B(0, R)$. Then $g(x)$ has the mean value property,

$$\ln|y| = g(0) = \frac{1}{2\pi R} \int_{|x|=R} g(x)dS_x$$

Therefore,

$$\frac{1}{2\pi} \int_{|x|=R} \ln|y - x|dS_x = R\ln|y|, \quad \text{for } |y| > R \quad (17)$$

For $|y| \leq R$, in particular take $y = 0$,

$$(V_\Delta\phi)(0) = \frac{1}{2\pi} \int_{|x|=R} \ln|x|dS_x = R\ln R$$

The relation (17) implies that, the limit of the value of the potential when $|y|$ approach the boundary from exterior is given by

$$\lim_{|y| \rightarrow R^+} (V_\Delta\phi)(y) = R\ln R \quad \text{for } |y| = R$$

Furthermore, since the single layer potential is continuous on \mathbb{R}^2 we have

$$(V_\Delta\phi)(y) = R\ln R \quad \text{for } |y| = R$$

To determine the value of the potential inside the disc for $y \neq 0$, we use the maximum/minimum principle. Since the single layer potential is harmonic on Ω it has neither maximum nor minimum in the disc. Let

$$C_0 = (V_\Delta\phi)(y_0) \quad \text{for } 0 < |y_0| < R$$

If we assume $C_0 \neq R \ln R$, i.e., C_0 is different from the value of potential on the boundary, we will arrive contradiction of the maximum principle. Thus $(V_\Delta \phi)(y)$ is constant on $\bar{\Omega}$. Therefore, $(V_\Delta \phi)(y) = R \ln R$, for $|y| \leq R$.

Remark 2. In the above example, if we take the value of $R = 1$, and since $a(y) \neq 0$, then $(V\phi)(y) = 0$ in $\bar{\Omega}$.

Example 1 shows that, the kernel of the operator $\mathcal{V}: H^{-1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$ contains non zero element for a unit ball, i.e., $\ker \mathcal{V} \neq \{0\}$ for $\Omega = B(0,1)$, which means, the operators \mathcal{V} is not one to one for this particular domain. Consequently, the following question may arise: does the kernel of \mathcal{V} contain a non-zero element on every bounded domain in \mathbb{R}^2 ? The answer is no.

Theorem 3. The following spaces are subspaces of $L_2(\rho; \Omega)$, $\mathcal{H}^{1,0}(\Omega; A)$ and $\tilde{H}^s(\Gamma_1)$, $H^s(\partial\Omega)$, respectively, Where $\Gamma_1 \subset \partial\Omega$

- (i) $L_2(\rho; \Omega) \perp \mathfrak{R} = \{f \in L_2(\rho; \Omega): \langle f, 1 \rangle_\Omega = 0\}$
- (ii) $\mathcal{H}^{1,0\perp}(\Omega; A) = \{g \in \mathcal{H}^1(\Omega): Ag \in L_2(\rho; \Omega) \perp \mathfrak{R}\}$
- (iii) $H_{**}^s(\partial\Omega) = \{\psi \in H^s(\partial\Omega): \langle \psi, 1 \rangle_{\partial\Omega} = 0\}$, $\tilde{H}_{**}^s(\Gamma_1) = \{\psi \in \tilde{H}^s(\Gamma_1): \langle \psi, 1 \rangle_{\Gamma_1} = 0\}$

Proof. (i) let f and g be in $L_2(\rho; \Omega) \perp \mathfrak{R}$ and $\alpha, \beta \in \mathfrak{R}$ then $\alpha f + \beta g \in L_2(\rho; \Omega)$ then

$$\begin{aligned} \langle \alpha f + \beta g, 1 \rangle_\Omega &= \langle \alpha f, 1 \rangle_\Omega + \langle \beta g, 1 \rangle_\Omega \\ &= \alpha \langle f, 1 \rangle_\Omega + \beta \langle g, 1 \rangle_\Omega \\ &= 0 \end{aligned}$$

(ii) let f and g be in $\mathcal{H}^{1,0\perp}(\Omega; A)$ and $\alpha, \beta \in \mathfrak{R}$ then $\alpha f + \beta g \in \mathcal{H}^{1,0}(\Omega; A)$ then by linearity of an operator A and $L_2(\rho; \Omega) \perp \mathfrak{R}$ above,

$$\begin{aligned} A(\alpha f + \beta g) &= A(\alpha f) + A(\beta g) \\ &= \alpha A f + \beta A g \\ &\in L_2(\rho; \Omega) \perp \mathfrak{R} \end{aligned}$$

(iii) let ψ and φ be in $H_{**}^s(\partial\Omega)$ and $\alpha, \beta \in \mathfrak{R}$ then $\alpha\psi + \beta\varphi \in H^s(\partial\Omega)$ then

$$\begin{aligned} \langle \alpha\psi + \beta\varphi, 1 \rangle_{\partial\Omega} &= \langle \alpha\psi, 1 \rangle_{\partial\Omega} + \langle \beta\varphi, 1 \rangle_{\partial\Omega} \\ &= \alpha \langle \psi, 1 \rangle_{\partial\Omega} + \beta \langle \varphi, 1 \rangle_{\partial\Omega} \\ &= 0 \end{aligned}$$

Similarly the right hand side of (iii) follows from the proof of item (iii).

In order to have invertibility for the single layer potential operator in $2D$, we consider the following theorem.

Theorem 4. If $\psi \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$ satisfies $\mathcal{V}\psi = 0$ on $\partial\Omega$, then $\psi = 0$.

Proof. The theorem holds for the operator \mathcal{V}_Δ (e.g, corollary 8.11(ii) in [9]),

$$\begin{aligned} \mathcal{V}_\psi &= 0 \\ \Rightarrow \frac{1}{a(y)} \mathcal{V}_\Delta \psi &= 0 \\ \Rightarrow \psi &= 0, \text{ (since } a(y) \neq 0, \Rightarrow \mathcal{V}_\Delta \neq 0) \end{aligned}$$

Lemma 2. If $u \in \mathcal{H}^{1,0\perp}(\Omega; A)$ then $T^+u \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$.

Proof. Employing the first Green identity (5) with $v = 1$, we have:

$$\begin{aligned} \langle T^+u, 1 \rangle_{\partial\Omega} &= \int_\Omega 1 A u dx \\ &= \langle Au, 1 \rangle_\Omega \\ &= 0; \text{ since } Au \in L_2(\rho; \Omega) \perp \mathfrak{R} \end{aligned}$$

In addition to conditions (3) and (4) on the coefficient a , we will sometimes also need the condition

$$\rho^2 \Delta a \in L_\infty(\mathbb{R}^2) \quad (18)$$

Employing that the corresponding mapping properties hold true for the potentials associated with the Laplace operator Δ , (eg. Section 8 in [14]) and references therein, relations (16) lead to the following assertion. (e.g., Theorem 4.1 in [4] and Theorem 3 in [19]).

Theorem 5. The following operators are continuous under conditions (4).

$$\begin{aligned} P: \mathcal{H}^{-1}(\mathbb{R}^2) \perp \mathfrak{R} &\rightarrow \mathcal{H}^1(\mathbb{R}^2) \\ \mathcal{P}: \tilde{\mathcal{H}}^{-1}(\Omega) \perp \mathfrak{R} &\rightarrow \mathcal{H}^1(\mathbb{R}^2) \\ R: L_2(\rho^{-1}; \mathbb{R}^2) &\rightarrow \mathcal{H}^1(\mathbb{R}^2) \\ V: H_{**}^{-\frac{1}{2}}(\partial\Omega) &\rightarrow \mathcal{H}^1(\Omega) \\ W: H^{\frac{1}{2}}(\partial\Omega) &\rightarrow \mathcal{H}^1(\Omega) \end{aligned}$$

while The following operators are continuous under conditions (4) and (18).

$$\begin{aligned} \mathcal{P}: L_2(\rho; \Omega) \perp \mathfrak{R} &\rightarrow \mathcal{H}^{1,0}(\mathbb{R}^2; A) \\ \mathcal{R}: \mathcal{H}^1(\Omega) &\rightarrow \mathcal{H}^{1,0}(\Omega; A) \\ V: H_{**}^{-\frac{1}{2}}(\partial\Omega) &\rightarrow \mathcal{H}^{1,0}(\Omega; A) \\ W: H^{\frac{1}{2}}(\partial\Omega) &\rightarrow \mathcal{H}^{1,0}(\Omega; A) \end{aligned}$$

Remark 3. Similar to Theorem 3.12 [11] one can prove that $\mathcal{D}(\bar{\Omega})$ is dense in $\mathcal{H}^{1,0}(\Omega; A)$ also in $\mathcal{H}^{1,0\perp}(\Omega; A)$ which then implies by theorem 5 and lemma 2, (13) holds for any $u \in \mathcal{H}^{1,0\perp}(\Omega; A)$.

The boundary integral (pseudo-differential) operators of the direct values and of the co-normal derivatives of the single and double layer potentials are defined by

$$\mathcal{V}g(y) := - \int_\Gamma P(x, y) g(x) ds_x;$$

$$\mathcal{W}g(y) := - \int_\Gamma T_x P(x, y) g(x) ds_x \quad y \in \Gamma \quad (19)$$

$$\mathcal{W}'g(y) := - \int_\Gamma T_y P(x, y) g(x) ds_x;$$

$$\mathcal{L}^\pm g(y) := T_y^\pm Wg(y) \quad y \in \Gamma \quad (20)$$

The mapping and jump properties of the operators (19)-(20) follow from relations (16) and are described in details in [18]. Particularly, their jump relations are given by the following theorem presented in Theorem 2, [18].

Theorem 6. let $g_1 \in H^{-\frac{1}{2}}(\Gamma)$, $g_2 \in H^{\frac{1}{2}}(\Gamma)$ and $a \in C^1(\mathbb{R}^2)$. Then

$$\begin{aligned}\gamma^+ V g_1(y) &:= \mathcal{V} g_1(y) \\ \gamma^+ W g_2(y) &:= \mp \frac{1}{2} g_2(y) + \mathcal{W} g_2(y) \\ T^\pm V g_1(y) &:= \pm \frac{1}{2} g_1(y) + \mathcal{W}' g_1(y)\end{aligned}$$

where $y \in \partial\Omega$.

employing the co-normal derivative and trace operators to the third Green identity (13), and using the jump relations for the potential operators we obtain for $u \in \mathcal{H}^{1,0\perp}(\Omega; A)$,

$$\frac{1}{2} \gamma^+ u + \gamma^+ \mathcal{R} u - \mathcal{V} T^+ u + \mathcal{W} \gamma^+ u = \gamma^+ \mathcal{P} A u \quad \text{on } \partial\Omega \quad (21)$$

$$\frac{1}{2} T^+ u + T^+ \mathcal{R} u - \mathcal{W}' T^+ u + \mathcal{L}^+ \gamma^+ u = T^+ \mathcal{P} A u \quad \text{on } \partial\Omega \quad (22)$$

Conditions (4) are assumed to hold for (21) and conditions (4) and (18) for (22). For some functions f, Ψ and Φ let us consider a more general indirect integral relation associated with equation (13).

$$u + \mathcal{R} u - V\Psi + W\Phi = \mathcal{P}_f \quad \text{in } \Omega \quad (23)$$

Lemma 3. Let $u \in \mathcal{H}^{1,0\perp}(\Omega)$, $f \in L_2(\rho; \Omega) \perp \mathfrak{R}$, $\Psi \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$, and $\Phi \in H^{\frac{1}{2}}(\partial\Omega)$ satisfy equation (23) and let conditions (4) and (18) hold. Then, u is a solution of the equation

$$A u = f \quad \text{in } \Omega \quad (24)$$

While

$$V(\Psi - T^+ u) - W(\Phi - \gamma^+ u) = 0, \quad \text{in } \Omega \quad (25)$$

Proof. Since $u \in \mathcal{H}^{1,0\perp}(\Omega; A)$, by Remark 3 we can write the third Green identity (13) for the function u . Then subtracting (23) from it, we obtain

$$-V\Psi^* + W\Phi^* = \mathcal{P}[A u - f] \quad \text{in } \Omega \quad (26)$$

where $\Psi^* := T^+ u - \Psi$ and $\Phi^* := \gamma^+ u - \Phi$. Multiplying equality (26) by $a(y)$ we get

$$-V_\Delta \Psi^* + W_\Delta(a\Phi^*) = \mathcal{P}_\Delta[A u - f] \quad \text{in } \Omega$$

Applying the Laplace operator Δ to the last equation and taking into consideration that both functions in the left-hand side are harmonic potentials, while the right-hand side function is the classical Newtonian potential, we arrive at Eq. (24)

Substituting (24) back into (26) leads to (25).

Lemma 4. Let conditions (4) and (18) hold.

(i) If $\Psi^* \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$ and $V\Psi^* = 0$ in Ω , then $\Psi^* = 0$.

(ii) If $\Phi^* \in H^{\frac{1}{2}}(\partial\Omega)$ and $W\Phi^*(y) = 0$ in Ω , then $\Phi^*(x) = C/a(x)$, where C is a constant.

(iii) let $\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$, where Γ_1 and Γ_2 are nonempty non intersecting simply connected submanifolds of $\partial\Omega$ with infinitely smooth boundaries.

If $\Psi^* \in \tilde{H}_{**}^{-\frac{1}{2}}(\Gamma_1)$, $\Phi^* \in \tilde{H}^{\frac{1}{2}}(\Gamma_2)$ and $V\Psi^*(y) - W\Psi^*(y) = 0$ in Ω , then $\Psi^* = 0$ and $\Phi^* = 0$ on $\partial\Omega$.

Proof. The proof of item (i) follows from theorem 4, while the proof of item (iii) is similar to the proof of Lemma 2.12 [20].

To prove item (ii), from the first Green identity (5) for the interior domain Ω^- employing for $v(x) = C$, $A = \Delta$, $u = \frac{\ln|x-y|}{2\pi}$ and for any $y \in \Omega$, the function $\Phi_\Delta = C$ satisfies the equation $W_\Delta \Phi_\Delta = 0$ in the exterior domain Ω for any $C = \text{const}$. Now let us check there is no other solution of the equation in Ω in $H^{\frac{1}{2}}(\partial\Omega)$. By the Lyapunov-Tauber theorem $T_\Delta^+ W_\Delta \Phi_\Delta = T_\Delta^- W_\Delta \Phi_\Delta = 0$ on $\partial\Omega$, which implies $W_\Delta \Phi_\Delta = \text{const}$ in the interior domain Ω^- due to the uniqueness up to a constant of the solution of the Neumann problem in $H^{\frac{1}{2}}(\Omega^-)$. Then by the jump property of the double layer $\Phi_\Delta = \text{const}$. Applying the relation $Wg = \frac{1}{a} W_\Delta(ag)$ completes the proof of item (ii).

6. BDIEs for Exterior Mixed BVP

To reduce the variable-coefficient Mixed BVP (7)-(9) to a segregated boundary domain integral equation systems, Let us fix an extension $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ of the given function φ_0 in the condition (8) from $\partial\Omega_D$ to the whole of $\partial\Omega$ and an extension $\Psi_0 \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$ of the given function ψ_0 in the condition (9) from $\partial\Omega_N$ to the whole of $\partial\Omega$. moreover Φ_0 and Ψ_0 are considered as known.

For a given function f in $L_2(\rho; \Omega) \perp \mathfrak{R}$, assume that the function u satisfies the PDE $Au = f$ in Ω . Then, we can reduce the BVP (7)-(9) to a system of Boundary-Domain Integral Equations (BDIEs) and in all of them we represent in (13), (21) and (22) the trace of the function u and in its co-normal derivative as

$$\gamma^+ u = \Phi_0 + \varphi, \quad \varphi \in \tilde{H}^{\frac{1}{2}}(\partial\Omega_N); \quad T^+ u = \Psi_0 + \psi, \quad \psi \in \tilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D)$$

and will regard the new unknown functions φ and ψ as formally segregated of $u \in \mathcal{H}^{1,0}(\Omega; A)$. Thus we will look for the triplet

$$\mathcal{U} = (u, \psi, \varphi)^\top := \mathcal{H}^{1,0}(\Omega; A) \times \tilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \tilde{H}^{\frac{1}{2}}(\partial\Omega_N)$$

BDIE system (M11). Obtained under conditions (4) and (18), using equation (13) in Ω , the restriction of equation (21) on $\partial\Omega_D$, and the restriction of equation (22) on $\partial\Omega_N$, we arrive at the BDIE system (M11) of three equations for the triplet of unknowns, (u, ψ, φ) ,

$$u + \mathcal{R}u - V\psi + W\varphi = F_0 \text{ in } \Omega \quad (27)$$

$$\gamma^+ \mathcal{R}u - \mathcal{V}\psi + \mathcal{W}\varphi = \gamma^+ F_0 - \Phi_0 \text{ on } \partial\Omega_D \quad (28)$$

$$T^+ \mathcal{R}u - \mathcal{W}'\psi + \mathcal{L}^+ \varphi = T^+ F_0 - \Psi_0 \text{ on } \partial\Omega_N \quad (29)$$

Where

$$F_0 := \mathcal{P}f + V\Psi_0 - W\Phi_0 \text{ in } \Omega \quad (30)$$

We denote the matrix operator of the left hand side of the systems (M11) as

$$\mathcal{M}^{11} := \begin{bmatrix} I + \mathcal{R} & -V & W \\ r_{\partial\Omega_D} \gamma^+ \mathcal{R} & -r_{\partial\Omega_D} \mathcal{V} & r_{\partial\Omega_D} \mathcal{W} \\ r_{\partial\Omega_N} T^+ \mathcal{R} & -r_{\partial\Omega_N} \mathcal{W}' & r_{\partial\Omega_N} \mathcal{L}^+ \end{bmatrix},$$

$$\mathcal{F}^{11} := \begin{bmatrix} F_0 \\ r_{\partial\Omega_D} \gamma^+ F_0 - \Phi_0 \\ r_{\partial\Omega_N} T^+ F_0 - \Psi_0 \end{bmatrix}$$

Remark 4. Due to the mapping properties of operators involved in \mathcal{M}^{11} , The operator $\mathcal{M}^{11}: \mathcal{H}^{1,0}(\Omega; A) \times \tilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \tilde{H}^{\frac{1}{2}}(\partial\Omega_N) \rightarrow \mathcal{H}^{1,0}(\Omega; A) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N)$ is bounded. And also $\mathcal{F}^{11} = 0$ if and only if $(f, \Phi_0, \Psi_0) = 0$.

Proof. (\Leftarrow) evidently true.

(\Rightarrow) from equation (30) we have that $F_0 \in \mathcal{H}^{1,0}(\Omega; A)$ and by our assumption $0 = F_0$ implies $F_0 \in \mathcal{H}^{1,0\perp}(\Omega; A)$, Lemma 3 with $F_0 = 0$ for u implies $f = 0$ and $V(\Psi_0) - W(\Phi_0) = 0$, in Ω and The equalities $\gamma^+ F_0 = \Phi_0$ on $\partial\Omega_D$ and $T^+ F_0 = \Psi_0$ on $\partial\Omega_N$, implies $\Phi_0 = \varphi_0 = 0$ on $\partial\Omega_D$ and $\Psi_0 = \psi_0 = 0$ on $\partial\Omega_N$ that is, $\Psi_0 \in \tilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D)$ and $\Phi_0 \in \tilde{H}^{\frac{1}{2}}(\partial\Omega_N)$. Lemma 4 (iii) implies $\Phi_0 = \Psi_0 = 0$.

BDIE system (M12). Obtained under conditions (4) and using equation (13) in Ω and equation (21) on the whole of $\partial\Omega$, we arrive at the BDIE system (M12) of two equations for the triplet (u, ψ, φ) ,

$$u + \mathcal{R}u - V\psi + W\varphi = F_0 \text{ in } \Omega$$

$$\frac{1}{2}\varphi + \gamma^+ \mathcal{R}u - \mathcal{V}\psi + \mathcal{W}\varphi = \gamma^+ F_0 - \Phi_0 \text{ on } \partial\Omega$$

The left hand side matrix operator of the system is

$$\mathcal{M}^{12} := \begin{bmatrix} I + \mathcal{R} & -V & W \\ \gamma^+ \mathcal{R} & -\mathcal{V} & \frac{1}{2}I + \mathcal{W} \end{bmatrix}, \mathcal{F}^{12} := \begin{bmatrix} F_0 \\ \gamma^+ F_0 - \Phi_0 \end{bmatrix}$$

Remark 5. Due to the mapping properties of operators involved in \mathcal{M}^{12} , The operator $\mathcal{M}^{12}: \mathcal{H}^{1,0}(\Omega; A) \times \tilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \tilde{H}^{\frac{1}{2}}(\partial\Omega_N) \rightarrow \mathcal{H}^{1,0}(\Omega; A) \times H^{\frac{1}{2}}(\partial\Omega)$ is bounded. And also $\mathcal{F}^{12} = 0$ if and only if $(f, \Phi_0, \Psi_0) = 0$.

Proof. (\Leftarrow) evidently true.

(\Rightarrow) from equation (30) we have that $F_0 \in \mathcal{H}^{1,0}(\Omega; A)$ and by our assumption $0 = F_0$ implies $F_0 \in \mathcal{H}^{1,0\perp}(\Omega; A)$, Lemma 3 with $F_0 = 0$ for u implies $f = 0$ and $V(\Psi_0) - W(\Phi_0) = 0$, in Ω and The equalities $\gamma^+ F_0 = \Phi_0$ on $\partial\Omega$, implies $\Phi_0 = 0$. Lemma 4 (i) implies $\Psi_0 = 0$.

BDIE system (M21). Obtained under conditions (4) and (18) and Using equation (13) in Ω and equation (22) on the whole of $\partial\Omega$, we arrive at the BDIE system (M21) of two equations for the triplet (u, ψ, φ) ,

$$u + \mathcal{R}u - V\psi + W\varphi = F_0 \text{ in } \Omega \quad (31)$$

$$\frac{1}{2}\psi + T^+ \mathcal{R}u - \mathcal{W}'\psi + \mathcal{L}^+ \varphi = T^+ F_0 - \Psi_0 \text{ on } \partial\Omega \quad (32)$$

The left hand side matrix operator of the system is

$$\mathcal{M}^{21} := \begin{bmatrix} I + \mathcal{R} & -V & W \\ T^+ \mathcal{R} & \frac{1}{2}I - \mathcal{W}' & \mathcal{L}^+ \end{bmatrix}, \mathcal{F}^{21} := \begin{bmatrix} F_0 \\ T^+ F_0 - \Psi_0 \end{bmatrix}$$

Remark 6. Due to the mapping properties of operators involved in \mathcal{M}^{21} , The operator $\mathcal{M}^{21}: \mathcal{H}^{1,0}(\Omega; A) \times \tilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \tilde{H}^{\frac{1}{2}}(\partial\Omega_N) \rightarrow \mathcal{H}^{1,0}(\Omega; A) \times H^{-\frac{1}{2}}(\partial\Omega)$ is bounded.

BDIE system (M22). Obtained under conditions (4) and (18) and using equation (13) in Ω , the restriction of equation (22) on $\partial\Omega_D$, and the restriction of equation (21) on $\partial\Omega_N$, we arrive for the triplet (u, ψ, φ) at the BDIE system (M22) of three equations of "almost" the second kind (up to the spaces),

$$u + \mathcal{R}u - V\psi + W\varphi = F_0 \text{ in } \Omega$$

$$\frac{1}{2}\psi + T^+ \mathcal{R}u - \mathcal{W}'\psi + \mathcal{L}^+ \varphi = T^+ F_0 - \Psi_0 \text{ on } \partial\Omega_D$$

$$\frac{1}{2}\varphi + \gamma^+ \mathcal{R}u - \mathcal{V}\psi + \mathcal{W}\varphi = \gamma^+ F_0 - \Phi_0 \text{ on } \partial\Omega_N$$

The matrix operator of the left hand side of the system (M22) takes form

$$\mathcal{M}^{22} := \begin{bmatrix} I + \mathcal{R} & -V & W \\ r_{\partial\Omega_D} T^+ \mathcal{R} & r_{\partial\Omega_D} \left(\frac{1}{2}I - \mathcal{W}'\right) & r_{\partial\Omega_D} \mathcal{L}^+ \\ r_{\partial\Omega_N} \gamma^+ \mathcal{R} & -r_{\partial\Omega_N} \mathcal{V} & r_{\partial\Omega_N} \left(\frac{1}{2}I + \mathcal{W}\right) \end{bmatrix},$$

$$\mathcal{F}^{22} := \begin{bmatrix} F_0 \\ r_{\partial\Omega_D} \{T^+ F_0 - \Psi_0\} \\ r_{\partial\Omega_N} \{\gamma^+ F_0 - \Phi_0\} \end{bmatrix}$$

Remark 7. Due to the mapping properties of operators involved in \mathcal{M}^{22} , The operator $\mathcal{M}^{22}: \mathcal{H}^{1,0}(\Omega; A) \times H_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times H^{\frac{1}{2}}(\partial\Omega_N) \rightarrow \mathcal{H}^{1,0}(\Omega; A) \times H^{-\frac{1}{2}}(\partial\Omega_D) \times H^{\frac{1}{2}}(\partial\Omega_N)$ is bounded. And also $\mathcal{F}^{22} = 0$ if and only if $(f, \Phi_0, \Psi_0) = 0$.

Proof. The proof follows in the similar way as in the Remark 4 proof.

7. Equivalence and Uniqueness Theorems

Theorem 7. Let $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$, $\psi_0 \in H_{**}^{-\frac{1}{2}}(\partial\Omega_N)$, $L_2(\rho; \Omega) \perp \mathfrak{R}$ and let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$ be some extensions of φ_0 and ψ_0 , respectively, and conditions (4) and (18) hold.

(i) If a function $u \in \mathcal{H}^{1,0\perp}(\Omega; A)$ solves the BVP (7)-(9), then the triplet (u, ψ, φ) , where

$$T^+u - \Psi_0 = \psi \in \tilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D), \gamma^+u - \Phi_0 = \varphi \in \tilde{H}^{\frac{1}{2}}(\partial\Omega_N) \quad (33)$$

solves the BDIE systems (M11), (M12), (M21) and (M22).

(ii) If a triplet $(u, \psi, \varphi) \in \mathcal{H}^{1,0\perp}(\Omega; A) \times \tilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \tilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ solves one of the BDIE systems (M11), (M12) or (M22), then this solution is unique and solves all the systems, including (M21), while u solves BVP (7)-(9) and relations (33) hold.

Proof. (i) immediately follows from the deduction of the BDIE systems (M11), (M12), (M21) and (M22).

(ii) Let a triplet $(u, \psi, \varphi)^T$ solve BDIE system (M11), (M12) or (M22). The hypotheses of Lemma 3 are satisfied for the first equation in BDIE system, implying that u solves PDE (7) in Ω , while the following equation holds:

$$V\Psi^* - W\Phi^* = 0 \text{ in } \Omega \quad (34)$$

where $\Psi^* = \Psi_0 + \psi - T^+u$ and $\Phi^* = \Phi_0 + \varphi - \gamma^+u$.

Suppose first that the triplet $(u, \psi, \varphi)^T$ solves BDIE system (M11). Taking trace of (27) on $\partial\Omega_D$ using the jump relations of Theorem 6, and subtracting (28) from it, we obtain

$$\gamma^+u = \varphi_0 \text{ on } \partial\Omega_D \quad (35)$$

i.e., u satisfies the Dirichlet condition (8). Taking the co-normal derivative of Eq. (27) on $\partial\Omega_N$, using the jump relations on Theorem 6 and subtracting Eq. (29) from it, we obtain

$$T^+u = \psi_0 \text{ on } \partial\Omega_N \quad (36)$$

i.e., u satisfies the Neumann condition (9). Hence u solves the mixed BVP (7)-(9).

Taking into account $\varphi = 0, \Phi_0 = \varphi_0$ on $\partial\Omega_D$ and $\psi = 0, \Psi_0 = \psi_0$ on $\partial\Omega_N$, (35) and (36) imply that the first equation in (33) is satisfied on $\partial\Omega_N$ and the second equation in (33) is satisfied on $\partial\Omega_D$. Thus we have $\Psi^* \in \tilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D)$ and $\Phi^* \in \tilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ in (34). Let $\Gamma_1 = \partial\Omega_D, \Gamma_2 = \partial\Omega_N$. Then Lemma 4 (iii) implies $\Psi^* = \Phi^* = 0$, which completes the proof of conditions in (33). Uniqueness of the solution to BDIE systems (M11) follows from (33) along with remark 4 and Theorem 2.

Finally, item (i) implies that triplet $(u, \psi, \varphi)^T$ solves also BDIE systems (M12), (M21) and (M22).

Similar arguments work if we suppose that instead of the BDIE systems (M11), the triplet $(u, \psi, \varphi)^T$ solves BDIE systems (M12) or (M22).

The situation with uniqueness and equivalence for system (M21) differs from the one for other systems and from its counterpart BDIE system (M21) in [20], particularly because item (ii) of Lemma 4 is different from its analog, Lemma 2.11 (ii) in [20]. This leads to the following assertion.

Theorem 8. Let $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$, $\psi_0 \in H_{**}^{-\frac{1}{2}}(\partial\Omega_N)$, $f \in L_2(\rho; \Omega) \perp \mathfrak{R}$ and let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$ be some extensions of φ_0 and ψ_0 , respectively, and conditions (4) and (18) hold.

(i) Homogeneous BDIE system (M21) admits only one linearly independent solution $(u^0, \psi^0, \varphi^0) \in \mathcal{H}^{1,0\perp}(\Omega; A) \times \tilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \tilde{H}^{\frac{1}{2}}(\partial\Omega_N)$, where u^0 is the solution of the mixed BVP

$$Au^0 = 0 \text{ in } \Omega \quad (37)$$

$$r_{\partial\Omega_D}\gamma^+u^0 = \frac{1}{a(x)} \text{ on } \partial\Omega_D \quad (38)$$

$$r_{\partial\Omega_N}T^+u^0 = 0 \text{ on } \partial\Omega_N \quad (39)$$

While

$$\psi^0 = T^+u^0, \varphi^0 = \gamma^+u^0 - 1/a(x) \text{ on } \partial\Omega \quad (40)$$

(ii) The non-homogeneous BDIE systems (M21) is solvable, and any its Solution

$(u, \psi, \varphi) \in \mathcal{H}^{1,0\perp}(\Omega; A) \times \tilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \tilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ can be represented as

$$u = \tilde{u} + Cu^0 \text{ in } \Omega \quad (41)$$

where \tilde{u} solves the BVP (7)-(8) and C is a constant, while

$$\psi = T^+\tilde{u} - \Psi_0 + C\psi^0, \varphi = \gamma^+\tilde{u} - \Phi_0 + C\varphi^0 \text{ on } \partial\Omega \quad (42)$$

Proof. Problem (37)-(39) is uniquely solvable in $\mathcal{H}^{1,0\perp}(\Omega; A)$ by Theorem 2. Consequently, the third Green identity (13) is applicable to u^0 , leading to

$$u^0 + \mathcal{R}u^0 - V\psi^0 + W\varphi^0 = 0 \text{ in } \Omega \quad (43)$$

with notations (40), if we take into account that $W(1/a(x)) = 0$ in Ω due to the second relation in (16) and the equality $W_\Delta 1 = 0$ in Ω (cf. the proof of Lemma 4(ii)). Taking the co-normal derivative of (43) and substituting the first equation of (40) again, we arrive at

$$\frac{1}{2}\psi^0 + T^+\mathcal{R}u^0 - \mathcal{W}'\psi^0 + \mathcal{L}^+\varphi^0 = 0 \text{ on } \partial\Omega \quad (44)$$

Equations (43)-(44) mean that the triplet (u^0, ψ^0, φ^0) solves the homogeneous BDIE system (M21).

To prove item (ii) and check that there exists only one linearly independent solution of the homogeneous BDIE system (M21), we proceed as follows. First, we remark that the solvability of the non-homogeneous system (M21) follows from the solvability of the BVP (7)-(8) in $\mathcal{H}^{1,0,1}(\Omega; A)$ and the deduction of system (M21).

Let now a triplet $(u, \psi, \varphi)^\top \in \mathcal{H}^{1,0,1}(\Omega; A) \times \tilde{H}_{**}^{-\frac{1}{2}}(\partial\Omega_D) \times \tilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ solve (generally non-homogeneous) BDIE system (M21). Take the co-normal derivative of equation (31) on $\partial\Omega$ and subtract it from equation (32) to obtain

$$\psi + \Psi_0 - T^+u = 0 \text{ on } \partial\Omega \quad (45)$$

Taking into account that $\psi = 0$ and $\Psi_0 = \psi_0$ on $\partial\Omega_N$, this implies that u satisfies condition (9).

Equations (31) and (30) and Lemma 3 with $\Psi = \psi + \Psi_0, \Phi = \varphi + \Phi_0$ imply that u is a solution of equation (7) and

$$V(\Psi_0 + \psi - T^+u) - W(\Phi_0 + \varphi - \gamma^+u) = 0 \text{ in } \Omega \quad (46)$$

Due to (45) the first term vanishes in (46), and by Lemma 4(ii) we obtain

$$\Phi_0 + \varphi - \gamma^+u = -C/a(x) \text{ on } \partial\Omega \quad (47)$$

where C is a constant. Taking into account that $\varphi = 0$ on $\partial_D\Omega$ and $\Phi_0 = \varphi_0$ on $\partial\Omega_D$, we conclude that u satisfies the Dirichlet condition

$$\gamma^+u = \varphi_0 + C/a(x) \text{ on } \partial\Omega_D \quad (48)$$

instead of (8). Introducing notation \tilde{u} by (41) in (45), (47) and (48) and taking into account (37)-(39) prove the claim of item (ii). The case $\varphi_0 = 0, \Phi_0 = 0, \psi_0 = 0, \Psi_0 = 0, f = 0$ leading to the homogeneous BDIE system (M21) also implies that \tilde{u} for this case satisfies homogeneous BVP (7)-(9) and thus $\tilde{u} = 0$ in (41) and (42) meaning that the triplet (u^0, ψ^0, φ^0) is the only linearly independent solution of the homogeneous BDIE system (M21). This completes the proof of item (i) and of the whole theorem.

8. Future Work

Future work on the BDIEs for Variable Coefficient Mixed BVP in 2D Unbounded Domain, will consider the Fredholm properties and invertibility of the corresponding BDIOs in weighted Sobolev spaces. and we will also consider the Direct segregated systems of BDIEs for the Neumann BVPs for a scalar second order divergent elliptic PDEs with a variable coefficient in an exterior two-dimensional domain. and we will again consider the equivalence of BDIE system to the original boundary value problems and the Fredholm properties and invertibility of the corresponding BDIOs are to be analyzed in weighted Sobolev spaces in the future.

9. Conclusion

In this paper, we have considered a second-order elliptic partial differential equation with a variable coefficient in a 2D Unbounded domain, in appropriate weighted Sobolev space. The right-hand side functions were from $L_2(\rho; \Omega) \perp \mathfrak{R}$ and the Mixed data from the space $H_{**}^{-\frac{1}{2}}(\partial\Omega_N)$ and $H^{\frac{1}{2}}(\partial\Omega_D)$. The BVP was reduced to four systems of Boundary Domain Integral Equations and their equivalence to the original BVP and Uniqueness property was shown.

The properties of a parametrix-based potential operator that contain logarithmic singularity were investigated. Unlike properties in 3D case, The single layer potential needs special consideration to be invertible, which is critical on this study.

Abbreviations

BDIE	Boundary Domain Integral Equation
MBVP	Mixed Boundary Value Problem
PDE	Partial Differential Equation
BVP	Boundary Value Problem
BDIO	Boundary Domain Integral Operator

Acknowledgments

First and foremost, I would like to express my deepest gratitude to Engineer Masresha Kuma, who has been the inspiration behind this work and my entire mathematical journey. I am also profoundly grateful to my advisor, Dr. Tsegaye Ayele, for his unwavering support, insightful guidance, and constructive feedback throughout this endeavour.

Additionally, I would like to extend my heartfelt thanks to Professor Sergay Mikailov for his invaluable contributions and mentorship. Lastly, I would like to acknowledge my student, Makriana Birhanu who has been the inspiration behind this work.

Author Contributions

Eshetu Seid Ahimed is the sole author. The author read and approved the final manuscript.

Conflicts of Interest

The author declares no conflicts of interest.

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