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Research Article

Characterization of a Large Family of Convergent Series That Leads to a Rapid Acceleration of Slowly Convergent Logarithmic Series

Joseph Gaskin*

Department of Mathematics and Physics, Southern University and A&M College, Baton Rouge, USA

Abstract

Logarithmic series are known to have a very slow rate of convergence. For example, it takes more than the first 20,000 terms of the sum of the reciprocals of squares of the natural numbers to attain 5 decimal places of accuracy. In this paper, I will devise an acceleration scheme that will yield the same level of accuracy with just the first 400 terms of that power series. To accomplish this, I establish a relationship between all monotonically decreasing sequence of positive terms whose sum converges, a positive number ρ and a differentiable function φ. Then, I use ρ and φ to define the $T_{\varphi,\rho}$ transformations on the partial sums of any convergent series. Furthermore, I prove that these $T_{\varphi,\rho}$ transformations yield a rapid rate of convergence for many slowly convergent logarithmic series. Finaly, I provide several examples on how to compute φ if one is given the convergent series of decreasing, positive terms.

Keywords

Series, Accelerators, Logarithmic, Convergence

1. Introduction

This paper is devoted to accelerating the convergence of an infinite series $\sum_{n=1}^{\infty} a_n = S < \infty$, that satisfies $\lim_{n\to\infty}\left|\frac{S-a_{n+1}}{S-a_n}\right| = 1$. We call such series logarithmic series. Note that if $a_{n+1} \geq a_n$ and if $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 1$, then the convergent series $\sum_{n=1}^{\infty} a_n$, is a logarithmic series [2, 10].

Throughout this paper, $f(x) = a_x$ shall denote a continuous, positive valued, and a monotonically decreasing function on $[1,\infty)$, that satisfies:

1. $\lim_{x\to\infty}\frac{f(x+1)}{f(x)} = 1$ and
2. $\int_1^{\infty} f(x)dx < \infty$.

If μ is a positive real number greater than 1, then we extend the traditional definition of the partial sum of an infinite series [15] to define the μ^{th} partial sum, $S(\mu)$, of $\sum_{n=1}^{\infty} a_n$ as follows:

$S(\mu) = a_1 + a_2 + \cdots a_{[\mu]} + (\mu - [\mu])a_{[\mu]+1}$, where $[\mu]$ is the greatest integer less than μ.

If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences converging to A and B respectively, such that $\lim_{n\to\infty}\left|\frac{a_n-A}{b_n-B}\right| = 0$, then we say that $\{a_n\}_{n=1}^{\infty}$ converges more rapidly than $\{b_n\}_{n=1}^{\infty}$.

In [11], S. Mukherjee et. al cite a well-known theorem by J.P. Delahaye and Germain-Bonne [5] which states that: it is

*Corresponding author: joseph.gaskin@sus.edu (Joseph Gaskin)

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impossible to construct a series accelerator which can accelerate the convergence of all convergent series.

In this paper, we show that, for a large group of convergent, logarithmic series, $\sum_{n=1}^{\infty} a_n$, there exists a transformation, T, on the partial sums [6, 7] of the series and an increasing function φ on $[1,\infty)$, such that

3. $\varphi(n) > n$,
4. $T(S(\varphi(n))) \to S$,
5. $\lim_{n\to\infty} \left|\frac{T(S(\varphi))-S}{S(\varphi(n))-S}\right| = 0$.

In [1] (see pages 11 and 12), Brezinski shows that if: $T_{\varphi,\rho}(S(n)) = S(\varphi(n)) + D_{\varphi(n)}$, then $\lim_{n\to\infty}\left(\frac{D_{\varphi(n)}}{S-S(\varphi(n))}\right) = 0$ is equivalent to $\lim_{n\to\infty}\left|\frac{T_{\varphi,\rho}(S(n))-S}{S(\varphi(n))-S}\right| = 0$.

$D_{\varphi(n)}$ is then called a perfect estimation of the error of $S(\varphi(n))$.

Thus, we shall prove that if $\{a_n\}_{n=1}^{\infty}$ is a sequence of decreasing positive terms such that:

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 1$ and $0 < \lim_{n\to\infty}\left(\frac{S(\varphi(n))-S(\varphi(n-1))}{a_n}\right) = \rho < 1$,

then $D_{\varphi(n)} = \left(\frac{S(n)-S(\varphi(n))}{1-\frac{1}{\rho}}\right)$ will be a perfect estimation of the error of $S(\varphi(n))$.

This result leads naturally to a class of series accelerators:

$$T_{\varphi,\rho}(S(n)) = S(\varphi(n)) + D_{\varphi(n)} = S(\varphi(n)) + \left(\frac{S(n)-S(\varphi(n))}{1-\frac{1}{\rho}}\right)$$

The $T_{\varphi,\rho}$ transformations are extensions of the T_{+m} [3] accelerators by Clark and Gray (see [4]) since ρ works for all real numbers belonging to the interval $(0,1)$.

Furthermore, we shall also establish an interesting relationship between ρ, a convergent series $\sum_{n=1}^{\infty} a_n$, and an equation:

$$\varphi'(x)f(\varphi(x)) = \rho f(x)$$

It is worthy to note that if $\varphi(n) = n^2$, then the transformation:

$$T_{\varphi,\rho}(S(n)) = \left(\frac{S(n)-\frac{1}{\rho}S(\varphi(n))}{1-\frac{1}{\rho}}\right)$$

can be used to accelerate the convergence of the series $\sum_{n=1}^{\infty}\frac{1}{n(\ln(n))^2}$. I am not aware of any other series accelerator capable of accelerating the convergence of this series.

2. Section 1

We shall begin this section with an interesting result which characterizes all decreasing sequences $\{a_n\}_{n=1}^{\infty}$, whose sum, $\sum_{n=1}^{\infty} a_n$, converges.

Theorem 1

Let f be a continuous, positive valued, and decreasing function on $[1,\infty)$. Then, for each $\rho \in (0,1)$, $\sum_{n=1}^{\infty} f(n)$ converges if and only if there exists a differentiable function φ on $[1,\infty)$, such that:

6. $\varphi(x) > x$, and
7. $\varphi'(x)f(\varphi(x)) = \rho f(x)$

Proof:

Suppose that $\sum_{n=1}^{\infty} f(n)$ converges. Then, $\int_1^{\infty} f(t)dt$ converges [8, 14]. Furthermore, there exists $F(x)$, a negative-valued antiderivative of $f(x)$ that satisfies

$\lim_{x\to\infty} F(x) = 0$ [9]. Let $\varphi(x) = F^{-1}(\rho F(x))$.

Differentiating $\varphi(x)$ with respect to x, we get $\varphi'(x)f(\varphi(x)) = \rho f(x)$.

Since $F(x)$ is increasing on $[1,\infty)$, $\rho F(x) > F(x)$.

Thus, $\varphi(x) = F^{-1}(\rho F(x)) > F^{-1}((F(x)) = x$.

Next, suppose that $\varphi(x) > x$, and that $\varphi'(x)f(\varphi(x)) = \rho f(x)$.

Then, $\frac{\varphi'(x)f(\varphi(x))}{f(x)} = \rho < 1$. Therefore, by [2] (see page 44), $\sum_{n=1}^{\infty} f(n)$ converges.

Before proving our next Theorem, we shall give a couple of definitions. These definitions will mainly serve the purpose of simplifying the statement of Theorem 2.

Definition 1

If φ is a function satisfying the following conditions:

8. $\varphi(x) > x$ and $\varphi'(x) \geq 1$, and if
9. $\lim_{x\to\infty}\frac{\varphi'(\xi_x)}{\varphi'(\eta_x)} = 1$, where ξ_x and $\eta_x \in [x, x+1]$,

then we say that φ is logarithmic on $[1,\infty)$.

Definition 2

Let $\varphi(x)$ be a positive valued function and let ρ be an arbitrarily chosen positive number lying in the interval $(0,1)$. We define the $T_{\varphi,\rho}$ transformations as follows:

$$T_{\varphi,\rho}(S(n)) = S(\varphi(n)) + \left(\frac{S(n)-S(\varphi(n))}{1-\frac{1}{\rho}}\right)$$

Theorem 2

Suppose that $\sum_{n=1}^{\infty} f(n)$ is a convergent, logarithmic series of a monotonically decreasing sequence of positive terms and that φ is logarithmic on $[1,\infty)$. Then, $T_{\varphi,\rho}(S(n)) = \left(\frac{S(n)-\frac{1}{\rho}S(\varphi(n))}{1-\frac{1}{\rho}}\right)$ converges more rapidly than $S(\varphi(n))$.

Proof:

First, we shall show that $\lim_{n\to\infty}\left(\frac{S(\varphi(n))-S(\varphi(n-1))}{f(n)}\right) = \rho$.

To this end, note that

$$\frac{S(\varphi(n))-S(\varphi(n-1))}{f(n)} \leq \frac{(\varphi(n))-\varphi(n-1))f(\varphi(n-1))}{f(n)}$$

$$\leq \frac{\varphi'(\xi_n)f(\varphi(n-1))}{f(n)}$$

$$\leq \left(\frac{\varphi'(n-1)f(\varphi(n-1))}{f(n-1)}\right)\left(\frac{f(n-1)}{f(n)}\right)\left(\frac{\varphi'(\xi_n)}{\varphi'(n-1)}\right)$$

$$\leq \rho\left(\frac{f(n-1)}{f(n)}\right)\left(\frac{\varphi'(\xi_n)}{\varphi'(n-1)}\right), \text{ where } \xi_n \in [n-1, n].$$

Thus, $\overline{lim}\frac{S(\varphi(n))-S(\varphi(n-1))}{f(n)} \leq \rho$.

Similarly,

$$\frac{S(\varphi(n))-S(\varphi(n-1))}{f(n)} \geq \frac{(\varphi(n))-\varphi(n-1))f(\varphi(n\_)}{f(n)}$$

$$\geq \left(\frac{\varphi'(\eta_n)f(\varphi(n))}{f(n)}\right)$$

$$\geq \left(\frac{\varphi'(n)f(\varphi(n))}{f(n)}\right)\left(\frac{\varphi'(\eta_n)}{\varphi'(n)}\right)$$

$$\geq \rho\left(\frac{\varphi'(\eta_n)}{\varphi'(n)}\right), \text{ where } \eta_n \in [n-1, n]$$

Therefore, $\underline{lim}\frac{S(\varphi(n))-S(\varphi(n-1))}{f(n)} \geq \rho$.

Hence, $\lim_{n\to\infty}\frac{S(\varphi(n))-S(\varphi(n-1))}{f(n)} = \rho$ [13].

Now, $S(\varphi(n)) - S(n))$ and $S(\varphi(n)) - S$ both converge to 0 monotonically.

Therefore, by [2] (see page 413) and above, we have that:

$$\lim_{n\to\infty}\frac{S(\varphi(n))-S(n)}{S(\varphi(n))-S} = 1 - \lim_{n\to\infty}\left(\frac{f(n)}{S(\varphi(n))-S(\varphi(n-1))}\right) = \frac{\rho-1}{\rho}$$

Hence, $\lim_{n\to\infty}\left(\frac{T_{\varphi,\rho}(S(n))-S}{S(\varphi(n))-S}\right)$

$$= \lim_{n\to\infty}\left(\frac{T_{\varphi,\rho}(S(n))-S(\varphi(n))+S(\varphi(n))-S}{S(\varphi(n))-S}\right)$$

$$=1 - \lim_{n\to\infty}\left(\frac{D_{\varphi(n)}}{S(\varphi(n))-S}\right)$$

$$= 1 - \left(\frac{1}{1-\frac{1}{\rho}}\right)\lim_{n\to\infty}\left(\frac{S(\varphi(n))-S(n)}{S(\varphi(n))-S}\right)$$

$$= 1 - \left(\frac{1}{1-\frac{1}{\rho}}\right)\left(1-\frac{1}{\rho}\right) = 0$$

Thus,

$$D_{\varphi(n)} = \left(\frac{S(n)-S(\varphi(n))}{1-\frac{1}{\rho}}\right)$$

is a perfect estimation of the error of $S(\varphi(n))$, whenever φ is logarithmic on $[1,\infty)$, and

$$T_{\varphi,\rho}(S(n)) = S(\varphi(n)) + D_{\varphi(n)}$$

Corollary 2.1

Suppose that $\sum_{n=1}^{\infty} f(n)$ is a convergent logarithmic series of a monotonically decreasing sequence of positive terms. If $\lim_{x\to\infty}\frac{f(\varphi(x+1))}{f(\varphi(x))} = 1$, then, $T_{\varphi,\rho}(S(n)) = \left(\frac{S(n)-\frac{1}{\rho}S(\varphi(n))}{1-\frac{1}{\rho}}\right)$ converges more rapidly than $S(\varphi(n))$.

Proof:

Note that $\frac{\varphi'(x+1)}{\varphi'(x)} = \frac{f(x+1)}{f(x)}.\frac{f(\varphi(x+1))}{f(\varphi(x))}$, and $\lim_{x\to\infty}\frac{f(x+1)}{f(x)} = 1$.

Next, we provide some examples showing how to determine $\varphi(x)$ for a few convergent logarithmic series.

3. Examples

Example 1. Let $f(x) = \frac{1}{x^2}$. Then, $F(x) = \frac{-1}{x}$ and therefore $F^{-1}(x) = \frac{-1}{x}$.

Hence, $\varphi(x) = \frac{x}{\rho}$. If $\rho = \frac{1}{m}$, where m is a positive integer, then the $T_{\varphi,\rho}$ series accelerators reduce to the T_{+m} transformations (see [4], page 268, definition 4.1).

Example 2. Let $f(x) = \frac{1}{x(\ln(x))^2}$. Then, $F(x) = \frac{-1}{lnx}$ and $F^{-1}(x) = e^{\frac{-1}{x}}$. Thus,

$\varphi(x) = x^{\frac{1}{\rho}}$. Note that φ is logarithmic on $[1,\infty)$.

Example 3. Let $f(x) = \frac{1}{xlnx(\ln(lnx))^2}$. Then, $F(x) = \frac{-1}{\ln(\ln(x)}$ and $F^{-1}(x) = e^{e^{\frac{-1}{x}}}$. Hence, $\varphi(x) = e^{(lnx)^{\frac{1}{\rho}}}$ and φ is logarithmic on $[1,\infty)$ since $\varphi'(x) = \left(\frac{(lnx)^{\frac{1-\rho}{\rho}}}{\rho x}\varphi(x)\right)$.

Now, $\sum_{k=1}^{\infty}\frac{1}{k^2}$ =1.644.93 (5 decimal places) [12].

In the following table, $S(n) = \sum_{k=1}^{n}\frac{1}{k^2}$ (5 decimal places). We observe that it will take more than the first 20,000 terms of the series to attain 5 decimal places of accuracy. However, if one applies the $T_{\varphi,\rho}$ series accelerators, one only needs at most the first 400 terms to achieve the same level of accuracy.

Table 1. Partial sums of $\sum_{k=1}^{\infty}\frac{1}{k^2}$.

n	$S(n)$	$S(2n)$	$T_{2n,.5=}\ 2S(2n)-S(n)$	$S(4n)$	$T_{4n,.25}=\ {(4S(n)-S(n))}/{3}$
10	1.54977	1.59616	1.64256	1.62024	1.64374
100	1.63498	1.63995	1.64491	1.64244	1.64492
200	1.63995	1.64244	1.64493	1.64369	1.64493
1000	1.64393	1.64443	1.64493	1.64468	1.64493
10000	1.64483	1.64488	1.64493	1.64491	1.64493
100000	1.64492	1.64493	1.64493	1.64493	1.64493

4. Conclusion

In conclusion, we have shown if $\sum_{n=1}^{\infty} a_n$ is a convergent logarithmic series and if the φ function associated with this series is logarithmic, then the convergence of $\sum_{n=1}^{\infty} a_n$ can be accelerated by the $T_{\varphi,n}$ series accelerators.

Consequently, any convergent logarithmic series that can be written as a linear combination of logarithmically convergent series such that the associated φ functions converge logarithmically, can be accelerated by the $T_{\varphi,\rho}$ series accelerators.

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Author Contributions

Joseph Gaskin is the sole author. The author read and approved the final manuscript.

Conflicts of Interest

The author declares no conflict of interest.

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Research Fields

Joseph Gaskin: Real Analysis, Sequences and series, Number Theory, Differential equations, Point Set Topology